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ON THE USE OF COORDINATE PERTURBATIONS IN THE SOLUTION OF PHYSICAL PROBLEMS

by

PHYLLIS ANN FOX
Research Assistant in Mathematics
September 1951 — June 1953

Project for Machine Methods of Computation
and Numerical Analysis

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MACHINE METHODS
of
COMPUTATION
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FOREWORD

This is the first Technical Report of the Project on Machine Methods of Computation. The Project is financed by contract N5ori60 of the Office of Naval Research. It is an outgrowth of the activities of the Institute Committee on Machine Methods of Computation, established in November 1950.

Purpose of the Project is (1) to integrate the efforts of all the departments and groups at MIT who are working with modern computing machines and their applications and (2) to train men in the use of these machines for computation and numerical analysis.

PERSONNEL OF THE PROJECT

FACULTY SUPERVISORS: COMMITTEE ON MACHINE METHODS OF COMPUTATION

Philip M. Morse, Chairman	Physics
Samuel H. Caldwell	Electrical Engineering
Herman Feshbach	Physics
Jay W. Forrester	Electrical Engineering
Francis B. Hildebrand	Mathematics
John A. Hrones	Mechanical Engineering
Chia-Chiao Lin	Mathematics

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The work described in this particular report was done in parallel with the machine calculation of spherical wave propagation, which is to be reported separately. It was carried out in the Department of Mathematics under the supervision of Professor Chia-Chiao Lin and was accepted as a thesis in partial fulfillment of the requirements for the degree of Doctor of Science at the Massachusetts Institute of Technology, 1953. The writer wishes to express great appreciation to her supervisor, Professor Chia-Chiao Lin, who suggested the problem, for valuable suggestions and assistance given throughout the development of this work.

INTRODUCTION

This study deals with possible improvements effected in solutions to physical problems by changes in the coordinate system of the problem. In particular, perturbation solutions to the problems are investigated with a view to improving the usual solution by allowing a perturbation of the independent variables as well as of the dependent variables. The technique has proved useful for wave propagation problems of a hyperbolic nature described in Chapter 2, but has been less successful for the elliptic problem of the flow past a thin airfoil discussed in Chapter 3.

The idea of perturbing the independent variable of a problem was used some time ago by Poincaré in finding the limit cycle of a nonlinear oscillator (20), and more recently (1949) M. J. Lighthill considered the method at some length in a most interesting paper (12). Lighthill considers various types of ordinary differential equations whose usual perturbation solutions are unsatisfactory, and also in this paper he deals briefly with nonlinear partial differential equations. In a later paper (11) he applies the technique of coordinate perturbation to correcting the usual solution to the problem of incompressible flow past a thin airfoil. In the usual solution which is based on the small slope of the airfoil profile a satisfactory answer is not obtained near the leading edge of a blunt nosed airfoil, but Lighthill found that a small constant shift of the coordinate system corrected the

first order solution, making it valid uniformly, and he implied that successive corrections could be made in the coordinates as the higher order perturbation solutions were found.

This problem was the initial impetus of our investigation, and it was planned to study the problem at greater length and in particular with reference to the case of compressible flow. However, as the study progressed it became clear that the perturbation series developed for the independent variables were not satisfactory. Actually the successive perturbation functions in these series, in order to correct the increasing order of singularities in the velocity functions near the leading edge, have to carry the singularities themselves, and in the singular region each term of the perturbation series becomes of the same instead of decreasing order. The results and the failure of the series near the leading edge are discussed in more detail in Chapter 3.

Nevertheless the hope remained that at least some sort of one-stage correction could be found for the case of compressible flow past a thin airfoil, since the small constant coordinate shift does effectively correct the incompressible case. Several lines of investigation were attempted and various types of coordinate correction were considered, but unfortunately none of them proved successful. For example a constant coordinate shift is no longer correct since it does not satisfy the more complicated equations for the compressible

case. Then as far as other types of functions are concerned one important aspect arising from the investigation should be remarked. Upon attempting series expansions based on a perturbation parameter, $\mathcal{E}$, in the form

$$x = x^{(0)}(X,Y) + \varepsilon x^{(1)}(X,Y) + \varepsilon^2 x^{(2)}(X,Y) + \ldots \tag{1.1}$$

where X and Y are new coordinates, it was found that functions of the type, $x^{(i)}$, appearing in (1.1) could not be obtained, but rather that due both to boundary conditions and to changes of order in the derivatives of the perturbation functions that the functions were of the form

$$x^{(i)}(X,Y,\mathcal{E}) \tag{1.2}$$

In other words the functions, $x^{(i)}$, did not fit the usual type of perturbation solution (1.1).

Actually such functions might be acceptable in a new frame of perturbation solutions of the type

$$x = x^{(0)}(X,Y,\mathcal{E}) + \varepsilon x^{(1)}(X,Y,\varepsilon) + \varepsilon^2 x^{(2)}(X,Y,\varepsilon) + \ldots \tag{1.3}$$

where the $\mathcal{E}$ necessary to fulfill the requirements of the problem is allowed to appear in the perturbation functions. This line of investigation has not been pursued, but might lead to new methods of problem solution and is perhaps worthy of some study.

Next, since it had not been possible to find real solution improvement for the elliptic case, the investigation turned to problems of a hyperbolic character. In particular

the nonlinear hyperbolic partial differential equations governing wave propagation dependent on one space variable and one time variable were considered. Problems of this sort for the cases of spherical and cylindrical waves have been discussed by Whitham [23], [24], in papers where a correction in one set of characteristics is advocated. That is in lieu of using the usual functions of a linearized characteristic variable in the form

$$f(x - c_o t) \quad , \qquad c_o = \text{speed of sound at infinity} \tag{1.4}$$

where the characteristics are straight and parallel in the (x,t)-plane, Whitham suggests a correction in the form of a new variable, z, which is to be constant along the true characteristics of the problem now to be solved by functions of the type $f(z)$. He has obtained interesting results from this approach, but a drawback of the method is its failure to deal with both sets of characteristics at the same time.

The question arose as to whether it might not be possible to devise a method correcting the solution along both characteristic directions, and thus allow for waves propagating in either direction. It seemed natural to consider solutions expanded in terms of characteristic variables both for the dependent and independent variables, since in this way both the physical quantities and the correct characteristic curves might be found in the physical (x,t)-plane by a mapping from the characteristic plane. Attention was restricted mainly to the case of plane wave propagation where the equations are

simpler than for higher dimensional cases, and a solution was attempted in terms of characteristic variables α and β for velocity, u, and speed of sound, c, and for the originally independent space and time variables x and t. For a perturbation parameter, ε, the solutions were assumed in the form

$$
\begin{aligned}
u &= u^{(0)}(\alpha,\beta) + \varepsilon u^{(1)}(\alpha,\beta) + \varepsilon^2 u^{(2)}(\alpha,\beta) + \ldots \\
c &= c^{(0)}(\alpha,\beta) + \varepsilon c^{(1)}(\alpha,\beta) + \varepsilon^2 c^{(2)}(\alpha,\beta) + \ldots \\
x &= x^{(0)}(\alpha,\beta) + \varepsilon x^{(1)}(\alpha,\beta) + \varepsilon^2 x^{(2)}(\alpha,\beta) + \ldots \\
t &= t^{(0)}(\alpha,\beta) + \varepsilon t^{(1)}(\alpha,\beta) + \varepsilon^2 t^{(2)}(\alpha,\beta) + \ldots
\end{aligned}
\tag{1.5}
$$

Such a type of solution proved to be most satisfactory. In the case of a plane wave the series for u and c terminate, and a general expression can be found for the nth perturbation function in the series for x and for t. Furthermore, and of some importance, one finds that the convergence of these latter two series can be demonstrated for sufficiently small initial disturbances so that the correctness of the solution is assured. More precisely, the convergence is dependent on the size of the initial disturbance u' and c' to the base quantities u_o and c_o only to the extent of requiring that

$$\frac{u'}{u_o} < \frac{2}{\gamma+1} \qquad \frac{c'}{c_o} < \frac{\gamma-1}{\gamma+1} \tag{1.6}$$

where γ is the adiabatic exponent so that for example if $\gamma = 1.4$

$$\frac{u'}{u_o} < 0.833 \quad , \qquad \frac{c'}{c_o} < 0.166 \ .$$

In terms of density, ρ, this becomes

$$\frac{\rho'}{\rho_0} < 1.161$$

which is of the order of the requirement on velocity disburbance. In other words the series is a convergent one even for an initial density ratio as high as 2.16.

The convergence of the series in the ordinary regions of the flow is useful, but the outstanding fact is that convergence persists into the region where the mapping from the characteristic plane onto the physical plane becomes multiple-valued and where a shock would develop. In such a region the characteristics in the physical plane form an envelope, and one can show from the solutions (1.5) that the envelope develops for t of the order of $1/\varepsilon$. Then by digressing to state that the usual type of perturbation solution takes the form (see Chapter 2)

$$u = f^{(0)}(x,t) + \varepsilon f^{(1)}(x,t) + \varepsilon^2 t f^{(2)}(x,t) + \varepsilon^3 t\, f^{(3)}(x,t) + \ldots$$

or (1.7)

$$u = f^{(0)}(x,t) + \varepsilon\left[f^{(1)}(x,t) + \varepsilon t f^{(2)}(x,t) + (\varepsilon t)^2 f^{(3)}(x,t) + \ldots\right]$$

the great improvement introduced by solution (1.5) becomes clear. Equation (1.5) converges in a perturbation sense of decreasing order of terms even as t becomes $O(1/\varepsilon)$, but the terms in (1.7) become all of the same order for this size t and no longer represent a satisfactory perturbation solution. The solution (1.7) fails in the same way as that for the elliptic case did, but (1.5) remains satisfactory.

In extending (1.5) into the region where the solution becomes multiple-valued, a shock wave is introduced to allow a jump in the physical quantities. Its location can be satisfactorily demonstrated, and one can show that the shock speed is the average of the slopes of the characteristics on either side. The distance, x, which the solution may be continued beyond the point of shock development depends on the degree of accuracy desired. It is shown that an error of the order of $\varepsilon x^{3/2}$ is introduced into the solution at a distance x along the shock.

The theory developed in Chapter 2 is illustrated there by a particular example of an initial value problem involving a periodic distribution of density in a fluid initially at rest. The solution to the problem is found both for the usual type of perturbation theory and for the improved type, and the advantages of the second type are demonstrated.

One would hope that the same advantages arising from coordinate perturbation in the plane wave case would carry over to the higher dimensional cases of cylindrical and spherical flow. Of course one can carry out the same steps of the process, but for these cases where the equations are more complicated the series for u and c no longer terminate, and as yet no convergence proof for the series has been obtained. There is thus no guarantee of the solutions, but as is after all usually the case with perturbation solutions one can assume validity for "small enough" epsilon. It seems appropriate to extend the coordinate per-

turbation technique to these problems, justifying the procedure by the improvement the technique brought to the plane wave case, and hoping that in time a convergence proof will be developed outlining the regions of validity of a solution.

In summary then coordinate perturbation is a very useful technique for improving the perturbation solutions to problems of a hyperbolic nature. The characteristic variables are the "natural" variables of the problem and it now seems clear that all the quantities of a problem including the physical coordinates should be expanded in terms of the characteristic variables if the best type of perturbation scheme is to be used. For the elliptic case on the other hand no set of natural variables is available. Problem solution in terms of the imaginary characteristics is of no advantage, and so when a solution becomes unsatisfactory and singularities arise one's only recourse is to modify their effect in some way. In the case of the airfoil the singularity could be partially hidden inside the profile but its presence was still felt in the region of the leading edge and the singular behavior of the solution in the region persisted.

It is concluded that coordinate perturbation as discussed in this thesis does not yet offer a real improvement to elliptic problems, but rather finds great usefulness in its application to hyperbolic problems where the technique allows the solution to hold even somewhat after the development of a shock.

THE HYPERBOLIC CASE

A. Plane Wave Propagation

Introduction:

The solution to the problem of one-dimensional isentropic wave propagation may always be given in the form of an integral involving the Riemann function and the initial conditions and boundary conditions of the problem [16], [1]. However, except in certain special cases, such a representation of the solution is not very tractable, and some other form would be more useful.

One of the alternative forms of solution often appropriate, and in fact one of the favorite approaches of applied mathematicians, is that of a perturbation solution where the problem is linearized by expanding the solution in terms of a small parameter of the problem. The resultant solution is satisfactory as long as the series behaves properly, but there may be regions where the solution fails. In particular, as will be shown later, for the problem of plane wave propagation the series will diverge in regions where a shock starts to form and in these regions the series solution is no longer satisfactory.

In the attempt to overcome such drawbacks of the perturbation type of solution, and in line with the general aims of this investigation, a solution to the plane wave problem has been found in terms of a perturbation not only of the dependent quantities, velocity and density, but also of the space and time variables, x and t. The new independent variables

used are the characteristic parameters, and the four perturbation series for x, t, velocity, and density are expanded in terms of functions of the characteristics. The solution seems most promising in that the convergence of the series can be proved for an appropriately chosen perturbation parameter, and also in the fact that the region of shock development can be studied. That is, even though the mapping of the characteristic plane onto the physical plane becomes multiple-valued indicating the appearance of a shock, the solution series does not break down, and may be used to describe the phenomenon even somewhat after the formation of a weak shock.

In the first section of this part the four perturbation series are found for the initial value problem with general initial conditions, and a proof of the convergence of the series is given. An example of a simple physical problem is then given to illustrate such aspects as the inadequacies of the usual type of perturbation solution and the improvement effected by the improved perturbation solution. A further simplification of the example problem is then made so that the solution may be found easily from the Riemann function, expanded in terms of the small perturbation parameter, ε , and compared with the perturbation solution. The solutions are found to agree and in fact the order of ε required to allow the expansion of this latter solution is that expected from the general theory.

The second section is concerned with the incidence of a shock wave. The geometry of the mapping of the characteristic plane onto the physical plane is discussed, and the ability of

the solution to penetrate slightly into the region where the mapping starts to "fold over" is demonstrated. In the discussion entropy changes are neglected so that accuracy only through second order in shock strength is maintained. The same physical example used above is described near the regions of shock development.

Section 1: Perturbation solution and proof of convergence.

When the equations governing the propagation of a plane wave are cast into the form of differential equations along the characteristic directions of the problem, they become, for the case of a polytropic gas,

$$
\begin{aligned}
x_\alpha &= (u + c)t_\alpha \qquad & \frac{u_\alpha}{2} + \frac{c_\alpha}{\gamma - 1} &= 0 \\
x_\beta &= (u - c)t_\beta \qquad & \frac{u_\beta}{2} - \frac{c_\beta}{\gamma - 1} &= 0
\end{aligned}
\tag{2.1}
$$

where γ is the adiabatic exponent, α and β, the characteristic variables, and u and c the local velocity and speed of sound respectively [4].

Since a perturbation form of the solution to the initial value problem is to be obtained, the initial conditions must be given as disturbances superimposed on some prior uniform state. The choice of values to be assigned to α and β on the initial line is governed only by the requirement that a correct parametric (for example single-valued) representation of the initial state is achieved since the equations (2.1) are homogeneous in α or β. For simplicity the initial line $t = 0$, in the characteristic plane will be chosen as $x = \alpha = \beta$.

Then for a perturbation parameter ε, and initial perturbations $c_0 g(x)$ and $c_0 f(x)$ to c and u, the initial conditions become

$$\text{on } \alpha = \beta \begin{cases} x = \alpha = \beta \\ t = 0 \\ c = c_0(1 + \varepsilon g(\alpha)) = c_0(1 + \varepsilon g(\beta)) \\ u = \varepsilon c_0 f(\alpha) = \varepsilon c_0 f(\beta), \end{cases} \tag{2.2}$$

where the unperturbed gas was assumed at rest with uniform density.

A solution to equations (2.1) under the conditions of (2.2) is to be attempted in the form of the perturbation series,

$$\begin{aligned} x &= x^{(0)} + \varepsilon x^{(1)}(\alpha,\beta) + \varepsilon^2 x^{(2)}(\alpha,\beta) + \dots \\ t &= t^{(0)} + \varepsilon t^{(1)}(\alpha,\beta) + \varepsilon^2 t^{(2)}(\alpha,\beta) + \dots \\ u &= u^{(0)} + \varepsilon u^{(1)}(\alpha,\beta) + \varepsilon^2 u^{(2)}(\alpha,\beta) + \dots \\ c &= c^{(0)} + \varepsilon c^{(1)}(\alpha,\beta) + \varepsilon^2 c^{(2)}(\alpha,\beta) + \dots \end{aligned} \tag{2.3}$$

This form of solution will linearize the problem and give sets of equations from which the higher order perturbation functions may be successively determined.

The terms free from ε in the equations and initial conditions yield the initial approximation

$$
\begin{aligned}
x^{(0)} &= \frac{\alpha+\beta}{2} \\
c_0 t^{(0)} &= \frac{\alpha-\beta}{2} \\
u^{(0)} &= 0 \\
c^{(0)} &= c_0 .
\end{aligned}
\tag{2.4}
$$

For the functions u and c, the boundary conditions together with the two equations of (2.1) homogeneous in u and c, show that in (2.3) one will obtain

$$u^{(k)} = 0 \quad \text{and} \quad c^{(k)} = 0 \quad \text{for } k > 1, \tag{2.5}$$

whereas for $u^{(1)}$ and $c^{(1)}$ one finds

$$
\begin{aligned}
\frac{u^{(1)}}{2} + \frac{c^{(1)}}{\gamma-1} &= r(\beta) \\
\frac{u^{(1)}}{2} - \frac{c^{(1)}}{\gamma-1} &= -s(\alpha)
\end{aligned}
\tag{2.6a}
$$

where the initial conditions require

$$
\begin{aligned}
r(\beta) &= c_0\left(\frac{f(\beta)}{2} + \frac{g(\beta)}{\gamma-1}\right) \\
-s(\alpha) &= c_0\left(\frac{f(\alpha)}{2} - \frac{g(\alpha)}{\gamma-1}\right)
\end{aligned}
\tag{2.6b}
$$

From these expressions and from equations (2.1), any perturbation function $x^{(k)}$ or $t^{(k)}$ is found from,

$$x^{(k)} = \frac{1}{2}\int_\beta^\alpha (u^{(1)}+c^{(1)})\frac{\partial t^{(k-1)}}{\partial\alpha}d\alpha \;-\; \frac{1}{2}\int_\beta^\alpha (u^{(1)}-c^{(1)})\frac{\partial t^{(k-1)}}{\partial\beta}d\beta$$

$$t^{(k)} = -\frac{1}{2c_0}\int_\beta^\alpha (u^{(1)}+c^{(1)})\frac{\partial t^{(k-1)}}{\partial\alpha}d\alpha \;-\; \frac{1}{2c_0}\int_\beta^\alpha (u^{(1)}-c^{(1)})\frac{\partial t^{(k-1)}}{\partial\beta}d\beta \;. \qquad (2.7)$$

In equations (2.7) using equations (2.6) the integrands contain,

$$\frac{u^{(1)}+c^{(1)}}{2c_0} = Ar(\beta) - Bs(\alpha)$$

$$\frac{u^{(1)}-c^{(1)}}{2c_0} = Br(\beta) - As(\alpha) \qquad (2.8)$$

where $\quad A = \dfrac{\gamma+1}{4c_0}, \qquad B = \dfrac{3-\gamma}{4c_0}$.

Now since the functions $x^{(k)}$ and $t^{(k)}$ are analogous, only the functions $t^{(k)}$ will be treated, and it will be shown that these functions may be expressed in a certain general form. Then from this expression, the convergence of the perturbation series for t may be proved.

The form assumed by the functions will be shown to be,

$$c_0 t^{(n+1)} = \sum_{\lambda+\mu+\rho+\nu=n+1} R^{(n+1)\,\lambda,\mu}_{\rho,\nu}\; r^\rho(\beta)\, s^\nu(\alpha) \int_\beta^\alpha r^\lambda(\xi)\, s^\mu(\xi)\, d\xi \qquad (2.9)$$

where

$R^{(n+1)\,\lambda,\mu}_{\rho,\nu} = R^{(n+1)\,\mu,\lambda}_{\nu,\rho}$ = constants such that:

$$R^{(n+1)\,\lambda,\mu}_{\rho,\nu} = R^{(n)\,\lambda,\mu}_{\rho-1,\nu}\left(B\frac{\rho-1}{\rho} - A\right) + R^{(n)\,\lambda,\mu}_{\rho,\nu-1}\left(B\frac{\nu-1}{\nu} - A\right) + B\sum_{g=0}^{\mu-1}\frac{1}{g+1}R^{(n)\,\lambda,\mu-1-g}_{\rho,g}\,\delta_{\nu o} + B\sum_{g=0}^{\lambda-1}\frac{1}{g+1}R^{(n)\,\lambda-1-g,\mu}_{g,\nu}\,\delta_{\rho o}\,. \tag{2.10}$$

$\delta_{\nu o}$ = kronecker delta

(all indices in (2.10) non-negative)

Proof of the validity of (2.9) is carried out by mathematical induction from equations (2.7) and (2.8).

At the first stage for $\lambda + \mu + \rho + \nu = 0$ evidently (2.9) holds since

$$c_o t^{(0)} = \frac{\alpha - \beta}{2}$$

From (2.9):

$$\frac{\partial}{\partial\alpha}(c_o t^{(n)}) = \sum_{j+k+p+q=n} R^{(n)\,j,k}_{p,q}\, r^{p}(\beta)\left[q\, s^{q-1}(\alpha)\frac{\partial s}{\partial\alpha}\int_{\beta}^{\alpha} r^{j}(\mu)s^{k}(\mu)\,d\mu + s^{q+k}(\alpha)\, r^{j}(\alpha)\right]$$

$$\frac{\partial}{\partial\beta}(c_o t^{(n)}) = \sum_{j+k+p+q=n} R^{(n)\,j,k}_{p,q}\, s^{q}(\alpha)\left[p\, r^{p-1}(\beta)\frac{\partial r}{\partial\beta}\int_{\beta}^{\alpha} r^{j}(\mu)s^{k}(\mu)\,d\mu - r^{p+j}(\beta)\, s^{k}(\beta)\right] \tag{2.11}$$

so that equation (2.7) for $t^{(n+1)}$ becomes

$$C_0 t^{(n+1)} = \int_\beta^\alpha \sum_{j+k+p+q=n} R^{(n)\,j,k}_{p,q}\, r^p(\beta)\Big[B q s^q(\xi)\frac{\partial s}{\partial \xi}\int_\beta^\xi r^j s^k d\mu + B s^{q+k+1}(\xi)\, r^j(\xi) - $$
$$ - A q\, r(\beta)\, s^{q-1}(\xi)\frac{\partial s}{\partial \xi}\int_\beta^\xi r^j s^k d\mu - A r(\beta)\, s^{j+k}(\xi)\, r^j(\xi)\Big] d\xi +$$
$$+ \int_\beta^\alpha \sum_{j+k+p+q=n} R^{(n)\,j,k}_{p,q}\, s^q(\alpha)\Big[A p\, s(\alpha)\, r^{p-1}(\xi)\frac{\partial r}{\partial \xi}\int_\xi^\alpha r^j s^k d\mu - A s(\alpha)\, r^{p+j}(\xi)\, s^k(\xi) -$$
$$ - B p\, r^p(\xi)\frac{\partial r}{\partial \xi}\int_\xi^\alpha r^j s^k d\mu + B r^{p+j+1}(\xi)\, s^k(\xi)\Big] d\xi \; . \quad (2.12)$$

Then use is made of the formula,

$$\int_\beta^\alpha \frac{\partial s^r(\xi)}{\partial \xi}\left[\int_\beta^\xi r^m(\mu)\, s^n(\mu)\, d\mu\right] d\xi = s^r(\alpha)\int_\beta^\alpha r^m(\mu)\, s^n(\mu)\, d\mu - \int_\beta^\alpha r^m(\mu)\, s^{r+n}(\mu)\, d\mu \quad (2.13)$$

and equation (2.12) may be integrated using (2.13) to give

$$C_0 t^{(n+1)} = \sum_{j+k+p+q=n} R^{(n)\,j,k}_{p,q}\, r^p(\beta)\Big[\frac{Bq}{q+1} s^{q+1}(\alpha)\int_\beta^\alpha r^j s^k d\mu - \frac{Bq}{q+1}\int_\beta^\alpha r^j s^{k+q+1} d\mu - A r(\beta)\, s^q(\alpha)\int_\beta^\alpha r^j s^k d\mu +$$
$$+ A r(\beta)\int_\beta^\alpha r^j s^{k+q} d\mu + B\int_\beta^\alpha s^{q+k+1} r^j d\mu - A r(\beta)\int_\beta^\alpha s^{q+k} r^j d\mu\Big] +$$
$$+ \sum_{j+k+p+q=n} R^{(n)\,j,k}_{p,q}\, s^q(\alpha)\Big[\frac{Bp}{p+1} r^{p+1}(\beta)\int_\beta^\alpha r^j s^k d\mu - \frac{Bp}{p+1}\int_\beta^\alpha r^{j+p+1} s^k d\mu - A s(\alpha)\, r^p(\beta)\int_\beta^\alpha r^j s^k d\mu +$$
$$+ A s(\alpha)\int_\beta^\alpha r^{p+j} s^k d\mu + B\int_\beta^\alpha r^{p+j+1} s^q d\mu - A s(\alpha)\int_\beta^\alpha r^{p+j} s^k d\mu\Big] .$$

(2.14)

This expression is of the form (2.9) and upon proper treatment of indices yields the recurrence formula, (2.10). From (2.10) of course one finds that the rule for symmetry in the indices (by pairs $-(\lambda,\rho)$ and (μ,ν)) holds at the $(n+1)^{st}$ stage if it holds at the n^{th}.

The next step in the investigation is the proof of the convergence of the series for t,

$$t = t^{(0)} + \varepsilon t^{(1)} + \varepsilon^2 t^{(2)} + \ldots \tag{2.15}$$

from the expression (2.9) for the functions $t^{(k)}$.

It will be shown that if the functions r and s are bounded for the entire range of their argument by, say, a constant, M,

$$\begin{aligned} |r| &\leq M \\ |s| &\leq M, \end{aligned} \tag{2.16}$$

(which means of course from (2.6b) that the original perturbation functions f and g must be bounded), then the following geometric series may be shown to dominate (2.15),

$$\tfrac{1}{2}(\alpha - \beta) \sum_{j=0} \varepsilon^j \cdot (4A)^j \cdot M^j. \tag{2.17}$$

This series converges if ε is chosen such that

$$\varepsilon < \frac{1}{4AM}. \tag{2.18}$$

For proof, the terms in the series for $c_o t$ are evaluated from (2.9):

$$\left|c_0 t^{(n+1)} \varepsilon^{n+1}\right| \leq \varepsilon^{n+1} \left| \sum_{\lambda+\mu+\rho+\nu=n+1} R^{(n+1)\,\lambda,\mu}_{\qquad\rho,\nu} \; M^{\rho+\nu+\lambda+\mu} \; (\alpha-\beta) \right|$$

or (2.19)

$$\left|c_0 t^{(n+1)} \varepsilon^{n+1}\right| \leq \varepsilon^{n+1} \; M^{n+1} \; (\alpha-\beta) \sum_{\lambda+\mu+\rho+\nu=n+1} \left| R^{(n+1)\,\lambda,\mu}_{\qquad\rho,\nu} \right| .$$

The problem then resolves itself into estimating the expression

$$\sum_{\lambda+\mu+\rho+\nu} \left| R^{(n+1)\,\lambda,\mu}_{\qquad\rho,\nu} \right| . \tag{2.20}$$

The estimation is made by summing the recurrence formula (2.10) in terms of absolute values. Furthermore of course only three of the four indices λ, μ, ρ, ν are independent so one may set

$$\lambda = n+1-\mu-\rho-\nu . \tag{2.21}$$

Finally, to make use of the delta functions, the sum is broken up into the form

$$\sum_{m=0}^{n+1} \quad \sum_{\rho+\nu=m} \; \sum_{\mu=0}^{n+1-m} \equiv \sum_{m=0}^{n+1} \; \sum_{\nu=0}^{m} \; \sum_{\mu=0}^{n+1-m} \tag{2.22}$$

where $\rho = m - \nu$,

which represents correctly the requirement

$0 \leq \mu+\rho+\nu \leq n+1$, where no arrangement is

repeated.*

Then from (2.10)

$$\sum_{m=0}^{n+1}\sum_{\nu=0}^{m}\sum_{\mu=0}^{n+1-m}\left|R^{(n+1)\,n+1-\mu-m,\,\mu}_{m-\nu,\,\nu}\right| \le \sum_{m=0}^{n+1}\sum_{\nu=0}^{m}\sum_{\mu=0}^{n+1-m}\left(\left|R^{(n)\,n+1-\mu-m,\,\mu}_{m-\nu-1,\,\nu}\left(B\frac{m-\nu-1}{m-\nu}-A\right)\right| + \left|R^{(n)\,n+1-m-\mu,\,\mu}_{m-\nu,\,\nu-1}\left(B\frac{\nu-1}{\nu}-A\right)\right|\right) +$$

$$+\sum_{m=0}^{n+1}\sum_{\mu=0}^{n+1-m}\left(\sum_{g=0}^{\mu-1}\left|B\frac{1}{g+1}\,R^{(n)\,n+1-\mu-m,\,\mu-1-g}_{m,\,g}\right| + \sum_{g=0}^{n-m-\mu}\left|B\frac{1}{g+1}\,R^{(n)\,n-\mu-m-g,\,\mu}_{g,\,m}\right|\right). \tag{2.23}$$

In this expression, from (2.8)

$$A = \frac{\gamma + 1}{4c_0}$$

$$B = \frac{3 - \gamma}{4c_0}$$

so that for $\gamma > 1$, $A > B$ and

$$B\frac{\rho - 1}{\rho} - A \le A . \tag{2.24}$$

---

* To prove that
$$\sum_{m=0}^{n+1}\sum_{\rho+\nu=m}\sum_{\mu=0}^{n+1-m} R^{(n+1)\,n+1-\mu-m,\,\mu}_{\rho,\,\nu}$$
does include every type of R, it suffices to show that a particular $R^{(n+1)\,i,j}_{k,\ell}$ (i,j,k,ℓ = definite values such that $i + j + k + \ell = n+1$) can appear in one and only one way. In other words

m can only equal k
" " " $\ell \Rightarrow \nu \le m$ as required,
and " " " $j \Rightarrow \mu \le n+1-m$ as required.

Then, by replacing $\dfrac{1}{q+1}$ by 1 in (2.23), and by showing that in each of the form sums, expressions of the type,

$$\sum_{\lambda+\mu+\rho+\nu=n} \left| R^{(n)}\,{}^{\lambda,\mu}_{\rho,\nu} \right| \qquad \text{hold,}$$

(the proof in each case is the same as that at the bottom of page 19), one achieves an evaluation of (2.20)

$$\sum_{\lambda+\mu+\rho+\nu=n+1} \left| R^{(n+1)}\,{}^{\lambda,\mu}_{\rho,\nu} \right| \leq 4A \sum_{\lambda+\mu+\rho+\nu=n} \left| R^{(n)}\,{}^{\lambda,\mu}_{\rho,\nu} \right| . \tag{2.25}$$

So that if $T^{(n)}$ is an upper bound at the n^{th} stage,

$$\sum_{\lambda+\mu+\rho+\nu=n+1} \left| R^{(n+1)}\,{}^{\lambda,\mu}_{\rho,\nu} \right| \leq T^{(n+1)} \leq 4AT^{(n)} \leq \cdots \leq (4A)^{n+1} R^{(0)}\,{}^{0,0}_{0,0} = (4A)^{n+1}\frac{1}{2}$$

Finally then, from (2.19),

$$\left| c_o t^{(n+1)} \varepsilon^{n+1} \right| \leq \varepsilon^{n+1} M^{n+1} (\alpha - \beta\lambda(4A)^{n+1}.\ 1/2 \tag{2.26}$$

so that (2.17) does represent a dominant series for (2.15), requiring a restriction of ε of the form (2.18) for convergence, and the proof is completed.

Illustrative example

The points discussed so far in this section will be illustrated with reference to a particular simple example where the function $f(x)$ of equation (2.2) is taken to be zero, and the function $g(x)$ to be $\cos x$.

$$\text{Thus at } t = 0 \quad \begin{cases} u = 0 \\ c = c_o(1 + \varepsilon \cos x) \end{cases} \tag{2.27}$$

The general behavior of an initial periodic disturbance of this sort such as, for example, the development of the higher order harmonics and later the tendency of shocks to appear is of course of some interest in itself.

1. Usual perturbation solution

In the introduction it was mentioned that the usual type of perturbation solution in which only the dependent quantities u and c are expressed as perturbation series often is unsatisfactory. The particular way in which the solution is inadequate for this example problem will be demonstrated below.

The equations for plane wave propagation referred to x and t as independent variables are

$$\begin{aligned} c_t + uc_x + \frac{\gamma - 1}{2} cu_x &= 0 \\ cc_x + \frac{\gamma - 1}{2}(u_t + uu_x) &= 0. \end{aligned} \tag{2.28}$$

In the usual perturbation scheme a solution to (2.28) subject to the initial conditions of (2.27) is attempted in the form

$$\begin{aligned} c &= c^{(0)}(x,t) + \varepsilon c^{(1)}(x,t) + \varepsilon^2 c^{(2)}(x,t) + \ldots \\ u &= u^{(0)}(x,t) + \varepsilon u^{(1)}(x,t) + \varepsilon^2 u^{(2)}(x,t) + \ldots \end{aligned} \tag{2.29}$$

After the details have been worked out, the resulting perturbation functions become

$$\begin{aligned} c^{(0)} &= c_0 \\ u^{(0)} &= 0 \end{aligned} \tag{2.30}$$

$$c^{(1)} = \frac{c_o}{2}\left[\cos(x + c_o t) + \cos(x - c_o t)\right]$$

$$u^{(1)} = \frac{c_o}{\gamma - 1}\left[-\cos(x + c_o t) + \cos(x - c_o t)\right]$$

$$c^{(2)} = \frac{N}{2c_o}\left[\cos 2x - \cos 2c_o t + 1 - \tfrac{1}{2}\cos 2(x + c_o t) - \tfrac{1}{2}\cos 2(x - c_o t)\right] +$$

$$Mt\left[\sin 2(x + c_o t) - \sin 2(x - c_o t)\right]$$

$$u^{(2)} = \frac{N}{2c_o(\gamma - 1)}\left[\cos 2(x + c_o t) - \cos 2(x - c_o t)\right]$$

$$- \frac{2M}{\gamma - 1}\, t\left[\sin 2(x + c_o t) + \sin 2(x - c_o t)\right]$$

$$\text{where } M = -\frac{c_o^{\,2}}{4}\left(\frac{\gamma + 1}{2(\gamma - 1)}\right)$$

$$N = -\frac{c_o^{\,2}}{2}\left(\frac{\gamma - 3}{2(\gamma - 1)}\right)$$

$$c^{(3)} = a_1\cos 3(x + c_o t) + a_2\cos 3(x - c_o t) + a_3\cos(x + c_o t) +$$

$$a_4\cos(x - c_o t) + a_5\cos(3x + c_o t) +$$

$$a_6\cos(3x - c_o t) + a_7\cos(x + 3c_o t) + a_8\cos(x - 3c_o t) +$$

$$+ t\Big[b_1\sin 3(x + c_o t) + b_2\sin 3(x - c_o t) + b_3\sin(x + c_o t) +$$

$$b_4\sin(x - c_o t) + b_5\sin(3x + c_o t) + b_6\sin(3x - c_o t) +$$

$$b_7\sin(x + c_o t) + b_8\sin(x - 3c_o t)\Big] +$$

$$+ t^2\Big[c_1\cos 3(x + c_o t) + c_2\cos 3(x - c_o t) + c_3\cos(x + c_o t) +$$

$$c_4\cos(x - c_o t) + c_5\cos(3x + c_o t) + c_6\cos(3x - c_o t) +$$

$$c_7\cos(x + 3c_o t) + c_8\sin(x - 3c_o t)\Big] ,$$

where a_1, b_1, c_1 are constants determinable from the equations and initial conditions, and where the function $u^{(3)}$ has the same form with different constants.

$c^{(4)}$ has terms of the form:

constant, $\cos 2x$, $\cos 4x$, $\cos(2x \pm 2c_0 t)$, $\cos(2x \pm 4c_0 t)$, $\cos(4x \pm 2c_0 t)$, $\cos(4x \pm 4c_0 t)$, $\cos 4c_0 t$, $\cos 2c_0 t$,

$+\,t$ times sines of the same arguments,

$+\,t^2$ " " " " " "

$+\,t^3$ " " " " " " .

The inadequacy of such a solution appears as t becomes large, when the dominant terms in (2.29) become for example

$$c = c_0 + \varepsilon\left[c^{(1)} + \varepsilon t(\ ) + \varepsilon^2 t^2(\ ) + \ \dots\right]. \tag{2.31}$$

For $t = O(\frac{1}{\varepsilon})$ series (2.31) no longer converges in the perturbation sense, and in this sense the solution fails since every term in the bracketed expression becomes $O(1)$. The critical fact is that exactly this order of t holds at the region of shock development (see for example page 37) and thus the usual perturbation solution cannot hold into the region of shock development.

2. <u>Perturbation solution in characteristic coordinates</u>

If the same problem (initial conditions (2.27)) is solved by the method described earlier in this section, the perturbation functions are found as,

$$
\begin{aligned}
u^{(0)} &= 0 \\
c^{(0)} &= c_o \\
u^{(1)} &= \frac{c_o}{\gamma-1}(\cos\beta - \cos\alpha) \\
c^{(1)} &= \frac{c_o}{2}(\cos\beta + \cos\alpha) \\
&\left.\begin{aligned} u^{(j)} &= 0 \\ c^{(j)} &= 0 \end{aligned}\right\} \text{for } j > 1.
\end{aligned}
\tag{2.32}
$$

Thus

$$
\left.\begin{aligned}
u^{(1)} + c^{(1)} &= 2c_o(\bar{A}\cos\beta - \bar{B}\cos\alpha) \\
u^{(1)} - c^{(1)} &= 2c_o(\bar{B}\cos\beta - \bar{A}\cos\alpha)
\end{aligned}\right. , \text{ where } \begin{aligned} \bar{A} &= \frac{\gamma+1}{4(\gamma-1)} \\ \bar{B} &= \frac{3-\gamma}{4(\gamma-1)} \end{aligned}
$$

are to be used in finding

$$
\begin{aligned}
x &= x^{(0)} + \varepsilon x^{(1)} + \varepsilon^2 x^{(2)} + \dots \\
t &= t^{(0)} + \varepsilon t^{(1)} + \varepsilon^2 t^{(2)} + \dots ,
\end{aligned}
\tag{2.33}
$$

from the equations

$$
\begin{aligned}
x_\alpha &= (u + c)\, t_\alpha \\
x_\beta &= (u - c)\, t_\beta
\end{aligned}
\qquad \text{with } t = 0 \text{ on } x = \alpha = \beta .
$$

The resultant solutions are

$$
\begin{aligned}
x^{(0)} &= \frac{\alpha+\beta}{2} \\
c_o t^{(0)} &= \frac{\alpha-\beta}{2} \\
x^{(1)} &= \frac{\alpha-\beta}{2}(\cos\beta - \cos\alpha)\bar{A} \\
c_o t^{(1)} &= \frac{\alpha-\beta}{2}(-\cos\beta - \cos\alpha)\bar{A} + (\sin\alpha - \sin\beta)\bar{B} \\
x^{(2)} &= \frac{\alpha-\beta}{2}(\cos 2\alpha - \cos 2\beta)\left(\frac{\bar{A}^2}{2} + \frac{\overline{AB}}{4}\right) + \\
&\qquad 3\frac{\overline{AB}}{4}(2\cos\alpha\sin\beta + 2\cos\beta\sin\alpha - \sin 2\beta - \sin 2\alpha)
\end{aligned}
\tag{2.34}
$$

$$c_o t^{(2)} = \frac{\alpha-\beta}{2}(\cos 2\alpha + \cos 2\beta)\left(\frac{\bar{A}^2}{2} - \frac{\overline{AB}}{4}\right) + \frac{\alpha-\beta}{2}(\cos\alpha\cos\beta)2\bar{A}^2 +$$

$$\frac{\alpha-\beta}{2}(\bar{A}^2 + 2\bar{B}^2 - AB) + \text{terms not multiplied by } (\alpha-\beta).$$

$$x^{(3)} = \frac{\alpha-\beta}{2}(\cos 3\beta - \cos 3\alpha)(\quad)\text{ etc....}$$

$$\vdots$$

$$x^{(4)} = \frac{\alpha-\beta}{2}(\cos 4\beta - \cos 4\alpha)(\quad)\text{ etc...}$$

The form assumed by successive terms is apparent. In particular only the first power of $(\alpha-\beta)$ appears, and $\frac{\alpha-\beta}{2}$ is the initial approximation to t. It thus seems possible to continue the solution beyond the point where t becomes $O(\frac{1}{\varepsilon})$. That this is indeed possible is shown in section 2 where the details of the process are explored.

At this point it is of some interest to remark that since (2.32) and (2.34) constitute a correct solution to the problem, they must agree with the usual perturbation solution of (2.30) and also with the solution achieved using Riemann's function in all cases where the latter two solutions are obtainable and valid.

For the case of the usual perturbation solution the agreement may be demonstrated simply by introducing the coordinate perturbations of (2.33) and (2.34) into the solution (2.38) and showing that this causes the higher order perturbations $(j > 1)$ in u and c to become zero in agreement with (2.32). This has been checked for the cases of $u^{(2)}$ and $c^{(2)}$.

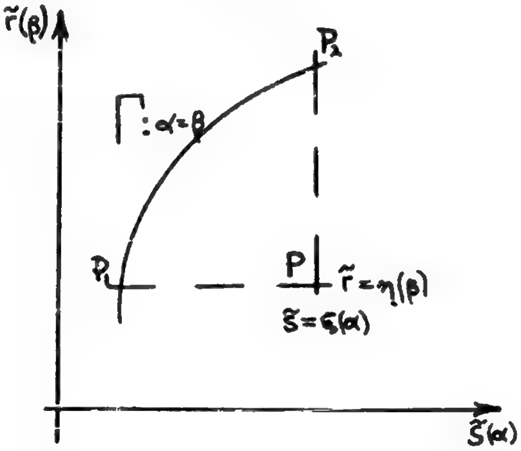
For the case where the solution is given in terms of Riemann's function, the solution may be shown to be (c.f. section 82 of [4])

$$t(\alpha,\beta) = t(\xi(\alpha),\eta(\beta)) = \int_{\Gamma} \frac{V(\lambda)}{2c(\lambda)}\,d\lambda,$$

where

$$c(\lambda) = c_o(1+\varepsilon g(\lambda)),$$

and where the point $\left\{\begin{array}{l}\tilde r = \eta(\beta)\\ \tilde s = \xi(\alpha)\end{array}\right\}$ is the point, P, in the $(\tilde r,\tilde s)$-plane of the figure corresponding to the point (α,β) in the (α,β)-plane.



The functions $\tilde r$ and $\tilde s$ on the curve Γ are

$$\tilde r(\alpha) = \frac{c_o}{\gamma-1} + \varepsilon r(\alpha)$$

$$\tilde s(\alpha) = \frac{c_o}{\gamma-1} + \varepsilon s(\alpha),$$

so that $\tilde r$ and $\tilde s$ represent the result of restoring the terms of order one to the perturbation functions, r and s, of equations (2.6b).

In the expression for $t(\alpha,\beta)$ the function $V(\lambda)$ is Riemann's function, and for this problem

$$V = \left(\frac{\xi+\eta}{\tilde r+\tilde s}\right)^{-1/2\mu^2} F\left(1-\frac{1}{2\mu^2},\ \frac{1}{2\mu^2},\ 1,\ z\right)$$

where F is the hypergeometric function and where

$$\mu^2 = \frac{\gamma-1}{\gamma+1}$$

$$z = -\frac{(\tilde s-\xi)(\tilde r-\eta)}{(\tilde s+\tilde r)(\xi+\eta)}\ .$$

Now if, at this stage, the problem is further simplified by assuming that $\frac{1}{2\mu^2}$ is an integer, the hypergeometric function, F, becomes a finite series. For example by choosing $\gamma = 1.4$, $\frac{1}{2\mu^2}$ becomes equal to 3, and gives

$$F = 1 - 6z + 6z^2.$$

Then using this F in the expression for $t(\alpha,\beta)$ and expanding the result in powers of ε such that

$$c_o t = c_o t^{(0)} + \varepsilon c_o t^{(1)} + \ldots.,$$

one obtains

$$c_o t^{(0)} = \frac{\alpha - \beta}{2}$$

$$c_o t^{(1)} = -3/2(\alpha - \beta)(\cos\alpha + \cos\beta) + (\sin\alpha - \sin\beta)$$

in agreement with the previous solution of (2.34) since for $\gamma = 1.4$

$$\bar{A} = \frac{\gamma + 1}{4(\gamma - 1)} = 3/2$$

$$\bar{B} = \frac{3 - \gamma}{4(\gamma - 1)} = 1.$$

Actually, so far as the details are concerned, the expansion in powers of ε as a perturbation series which arises out of terms of the sort

$$\frac{1}{\tilde{s}(\alpha) + \tilde{r}(\alpha)} = \frac{1}{\frac{2c_o}{\gamma - 1} + \varepsilon\left(r(\alpha) + s(\alpha)\right)}$$

is valid only so long as

$$\left|\varepsilon(r(\alpha) + s(\alpha)\right| \leq 2\varepsilon M < \frac{2c_o}{\gamma - 1},$$

where M is an upper bound on the functions, r and s.

For $A = \frac{\gamma+1}{4c_o}$ (see (2.8)) this means

$$\varepsilon < \frac{1}{4\mu^2 AM} .$$

Comparison of this inequality with the criterion of (2.18)

$$\varepsilon < \frac{1}{4AM} ,$$

shows that (2.18) does give a safe bound on ε, since for $\gamma > 1$

$$\frac{1}{\mu^2} > 1 .$$

What this requirement on ε really means in terms of the allowed initial departure of c from uniformity may be found by solving equations (2.6b) to find

$$c_o g(x) = \frac{\gamma-1}{2} (r(x) + s(x))$$

or

$$\varepsilon c_o |g| \leq (\gamma-1)\varepsilon M < \frac{\gamma-1}{4A} = (\frac{\gamma-1}{\gamma+1})c_o .$$

Thus the initial perturbation ratio for $\gamma = 1.4$ may be

$$\frac{\varepsilon c_o |g|}{c_o} < \frac{\gamma-1}{\gamma+1} = \frac{1}{6} ,$$

or, in terms of the disturbance, ρ', to the density ρ_o:

$$\frac{\rho'}{\rho_o} < 1.16,$$

as may be found from the relation

$$(1 + \frac{\rho'}{\rho_o}) = (1 + \frac{c'}{c_o})^{\frac{2}{\gamma-1}} .$$

Additional Example

As further evidence of the advantages accruing to the method of coordinate perturbation, another example will be considered.

The problem concerns a piston moving back and forth periodically in a semi-infinite cylinder. The velocity of the piston is given as $u_p = \varepsilon \sin \omega t$.

Now if the usual type of perturbation solution for u and c is obtained through the first order perturbations, and if the flux of mass past any given section is integrated over an integral number of periods of the motion,

$$\text{mass flux} = \int_0^T \rho u dt, \tag{2.35}$$

it will be found that there is a net mass flux outwards! Obviously this is in error, since there cannot be a continuous flow of matter away from the piston.

Actually, for one period, the mass flux average is:

$$\text{av. } \rho u = \frac{\varepsilon^2 \rho_0}{2c_0 \omega}, \quad \text{where } \begin{array}{l} \rho_0 = \text{av. density} \\ c_0 = \text{av. speed of sound.} \end{array} \tag{2.36}$$

That is, the quantity is of second order, and when the second order corrections to u and $c(\rho)$ have been obtained and included, the mass flux reduces correctly to zero. However, this additional computation required makes the task considerably more arduous. Also often in practice first order quantities are used to estimate say energy flux and so on (c.f. a recent

discussion and justification of this practice by A. Schock [17]), and in this problem of the piston the energy flux cannot be computed correctly from first order quantities; second order quantities are required to make the energy flux correctly balance the work done at the piston.

However, if the problem is described in terms of its characteristic variables, and if the variables x and t are also considered in perturbation form by the methods of this investigation, first order quantities will yield correct results for both mass and energy flux.

Briefly the results of the calculations are as follows.

Case i) Ordinary perturbation solution

A solution of the form

$$u = u^{(0)}(x,t) + \varepsilon u^{(1)}(x,t) + \varepsilon u^{(2)}(x,t) + \ldots$$

$$c = c^{(0)}(x,t) + \varepsilon c^{(1)}(x,t) + \varepsilon c^{(2)}(x,t) + \ldots$$

is sought for equations (2.28) under the boundary conditions that the velocity of the fluid at the piston be $u = \varepsilon \sin \omega t$, and that the solution contain only outward travelling waves.

The solution is:

$$\begin{aligned} u^{(0)} &= 0 \\ c^{(0)} &= 0 \\ u^{(1)} &= \sin\omega(t - \frac{x}{c_o}) \\ c^{(1)} &= \frac{\gamma-1}{2}\sin\omega(t - \frac{x}{c_o}) \end{aligned} \qquad (2.37)$$

$$c_o u^{(2)} = -\frac{1}{2} + \cos\omega(t - \frac{x}{c_o}) - \frac{1}{2}\cos 2\omega(t - \frac{x}{c_o}) + \frac{\gamma+1}{4}\frac{\omega}{c_o} x \sin 2\omega(t - \frac{x}{c_o})$$

$$c_o c^{(2)} = (\frac{\gamma-1}{2})\left[\cos\omega(t - \frac{x}{c_o}) - \frac{1}{2}\cos 2\omega(t - \frac{x}{c_o}) + \frac{\gamma+1}{4}\frac{\omega}{c_o} x \sin 2\omega(t - \frac{x}{c_o})\right].$$

For the mass flux, from (2.35)

$$\text{average mass flux} = \frac{1}{T}\int_0^T \rho u\,dt = \frac{1}{T}\int_0^T (\rho^{(0)} + \varepsilon\rho^{(1)})(u^{(0)} + \varepsilon u^{(1)})dt = \frac{\varepsilon^2 \rho_o}{2 c_o \omega}.$$

where $\frac{\rho}{\rho_o} = (\frac{c}{c_o})^{\frac{2}{\gamma-1}}$.

If second order terms are included:

$$\frac{1}{T}\int_0^T (\rho^{(0)} + \varepsilon\rho^{(1)} + \varepsilon^2\rho^{(2)})(u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)})dt = 0 + \text{order } \varepsilon^3.$$

For the flux at energy,

$$\text{given}\begin{cases} m = \rho u = \text{mass flux} \\ e = \text{internal energy} = C_v T = \frac{1}{\gamma-1}\frac{p}{\rho} \text{ (for the polytropic case)} \\ c^2 = \frac{\gamma p}{\rho} \\ p = \text{pressure} \end{cases}$$

the flux of energy is

$$m(\frac{1}{2}u^2 + e) + up = m(\frac{1}{2}u^2 + \frac{1}{\gamma-1}\frac{p}{\rho} + \frac{p}{\rho}) = m(\frac{1}{2}u^2 + \frac{1}{\gamma-1}c^2) \quad . \qquad (2.38)$$

The work done by the piston is the product pu at the piston.

Then, provided the second order terms in (2.37) are included, one may prove that

$$\left[\text{average energy flux at any point} = \text{average work done by piston} = \varepsilon^2 \frac{\rho_o c_o}{2}\right]. \qquad (2.39)$$

Case ii) Coordinate perturbation solution

A solution of the form (2.3) is sought for equations (2.1) under the conditions,

on the piston defined as $t = \alpha = \beta$:

$$
\begin{array}{lll}
x^{(0)} = 0 & t^{(0)} = \alpha = \beta & u^{(0)} = 0 \\
x^{(1)} = \frac{1}{\omega}(1 - \cos\omega t) & t^{(k)} = 0,\ k > 0 & u^{(1)} = \frac{dx^{(1)}}{dt} = \sin\omega t \\
x^{(k)} = 0,\ k > 1 & & u^{(k)} = 0,\ k > 1
\end{array} \tag{2.40}
$$

at the curve $\beta = 0$ which is the characteristic forming the boundary between the disturbed and undisturbed regions,

$$
\begin{array}{ll}
x^{(k)} = c_o t^{(k)} & u^{(k)} = 0, \text{ all } k \\
c^{(0)} = c_o & c^{(k)} = 0,\ k > 0.
\end{array} \tag{2.41}
$$

The solution is

$$
\begin{array}{ll}
u^{(0)} = 0 \quad , & c^{(0)} = c_o \\
u^{(1)} = \sin\omega\beta \quad , & c^{(1)} = \frac{\gamma-1}{2} \sin\omega\beta \\
u^{(k)} = 0 \quad , & c^{(k)} = 0,\ k > 1
\end{array}
$$

$$
\begin{aligned}
x^{(0)} &= \frac{c_o}{2}(\alpha - \beta) \\
c_o t^{(0)} &= \frac{c_o}{2}(\alpha + \beta) \\
x^{(1)} &= \frac{1}{\omega} + \frac{\gamma+1}{8}\left[(\alpha - \beta)\sin\omega\beta - \frac{1}{\omega}\cos\omega\alpha + \frac{\gamma-7}{\gamma+1}\frac{1}{\omega}\cos\omega\beta\right] \\
c_o t^{(1)} &= \frac{\gamma+1}{8}\left[-(\alpha - \beta)\sin\omega\beta - \frac{1}{\omega}\cos\omega\alpha + \frac{1}{\omega}\cos\omega\beta\right].
\end{aligned} \tag{2.42}
$$

The second order perturbations are not required.

(Incidentally a proof of the convergence of the perturbation series for this case has been worked out, but will not

be given since the details follow closely those of the proof for the initial value problem. For this boundary problem, given a case where the perturbation to the piston motion is bounded by a constant M, the requirement on ε is found to be

$$\varepsilon < \frac{4}{M}\frac{1}{(7 - \delta)}.) \qquad (2.43)$$

For case ii) the integrations for mass and energy flux must be carried out along the curve in the $(\alpha - \beta)$-plane which correctly represents a constant x location. When this is done, only the first order solutions of (2.42) need be used to predict zero mass flux and correct energy balance. A very definite saving in effort is effected, since the complexity of the computation greatly increases at the higher order perturbations.

Section 2: Shock development

In the solution to a problem of one-dimensional wave propagation a compression wave may appear which, steepening as it travels, forms ultimately a shock front. For the case where the shock develops in a simple wave region, Friedrichs has shown [6] that the solution may be continued with second order accuracy as a simple wave solution through the shock, so long as the shock is weak, since the neglected change in entropy is of third order in the shock strength.

We wish to show that a similar approach may be used when a shock forms in a region which is not a simple wave region,

but which needs both sets of characteristic variables for its description. In this section the type of perturbation solution in terms of characteristics found in the previous section will be studied from the point of view of its ability to describe the flow at the region of shock development and even somewhat beyond.

In general the development of a shock in a flow described by characteristic variables is characterized by the fact that the mapping from the characteristic plane to the physical plane ceases to be single valued; the image of the characteristic plane folds over to form a three-sheeted surface in the (x,t)-plane. At the edges of the fold the Jacobian,

$$J = x_{\alpha} t_{\beta} - x_{\beta} t_{\alpha} \tag{2.44}$$

is equal to zero, and within the region the velocity and density at any point (x,t) are triple-valued. This paradox is resolved by the introduction of a shock, since the jump in the values of velocity or density between the outermost sheets is condoned if a shock wave intervenes.

General descriptions of the geometry of the situation are to be found in Craggs [5], Stocker and Meyer [19], and Meyer [15]. The second problem of introducing a shock front in the correct way to restore the right solution has been studied usually for specific cases. As noted above, Friedrichs [4] has considered the simple wave case. Stocker [18] considered a particular situation involving a gas of adiabatic exponent, 5/3. Whitham [23], [24] has studied the development of shocks in the spherical

and axi-symmetrical cases by using coordinate perturbations of the type considered in this paper. However, he considers the nonlinear deviations of only one set of characteristics, and does not consider the other set, and thus really deals with the space analogue of a simple wave.

If, on the other hand, the deviation of both sets of characteristics from straight lines is considered, as is done in section 1, considerably more general types of problems should be solvable. In the following the series in terms of characteristic variables which were worked out for the case of a plane wave in section 1 are shown to converge even in the region where the mapping becomes multiple-valued. Then, just as for the case of simple waves, the series may be used to extend the solution beyond the point of shock wave incidence with an error depending only on neglect of entropy.

Development of the envelope of characteristics

In order to locate the region where the multiple-covering sets in, the Jacobian (2.44) is considered. Furthermore only the cases where the shock develops initially from one set of characteristic variables is treated. By this is meant simply that the second set of characteristics, which is treated in the fully nonlinear fashion, does not form an envelope at exactly the same time as does the first set. Specifically it is assumed that the multiple-valuedness occurs along the β -characteristics whose images in the (x,t)-plane form an envelope defined by

$$\frac{\partial x(\alpha,\beta)}{\partial \beta} = 0 \tag{2.45}$$

Then from equations (2.1), for $(u - c) \neq 0$,

$$\frac{\partial t(\alpha,\beta)}{\partial \beta} = 0, \tag{2.46}$$

and the Jacobian

$$J = x_\alpha t_\beta - x_\beta t_\alpha = 0, \tag{2.47}$$

so that the mapping of the (α,β)-plane into the (x,t)-plane is no longer one-to-one.

Then, given a perturbation solution of the type

$$c_o t = c_o t^{(0)} + \varepsilon c_o t^{(1)} + \varepsilon^2 c_o t^{(2)} + \dots,$$

(2.46) becomes

$$\frac{\partial c_o t}{\partial \beta} = \frac{\partial c_o t^{(0)}}{\partial \beta} + \varepsilon \frac{\partial c_o t^{(1)}}{\partial \beta} + \varepsilon^2 \frac{\partial c_o t^{(2)}}{\partial \beta} + \dots = 0 \tag{2.48}$$

(2.48) can hold only when the factor, $\dfrac{\partial (c_o t^{(2)})}{\partial \beta}$, of ε becomes of order $\frac{1}{\varepsilon}$ so that the second term of the series can balance the first. To see what this means in terms of the variables α and β, consider the perturbation functions given by the general form (2.9). For the first terms one finds:

$$c_o t = \frac{\alpha-\beta}{2} + \varepsilon\left[(\alpha-\beta)R^{(1)\,0,0}_{\;1,0}\left(r(\beta)+s(\alpha)\right) + R^{(1)\,1,0}_{\;0,0}\int_\beta^\alpha \left(r(\xi)+s(\xi)\right)d\xi\right] + \varepsilon^2\begin{pmatrix}\text{at most}\\ O(\alpha-\beta)M^2\end{pmatrix} + \dots$$

where from (2.16) the functions r and s are bounded by M.

Then

$$\frac{\partial c_0 t}{\partial \beta} = -\frac{1}{2} + \varepsilon \left[R^{(1)}{}^{0,0}_{1,0} \left((\alpha-\beta)\frac{\partial r(\beta)}{\partial \beta} - \left(r(\beta) + s(\alpha) \right) \right) - R^{(1)}{}^{1,0}_{0,0} \left(r(\beta) + s(\beta) \right) \right] +$$
$$+ \varepsilon^2 \left(\text{at most } O\left((\alpha-\beta) SM \right) \right), \qquad (2.49)$$

where
$$\left| \frac{\partial r(\mu)}{\partial \mu} \right| \leq S$$
$$\left| \frac{\partial s(\mu)}{\partial \mu} \right| \leq S$$
is required. (2.50)

So if the initial functions have derivatives satisfying (2.50) for the entire range of their arguments, one can have $\frac{\partial c_0 t}{\partial \beta} = 0$ only for

$$(\alpha - \beta) = O(\tfrac{1}{\varepsilon}), \qquad (2.51)$$

and a shock could be expected for large α and/or β corresponding to large t. The crucial fact involved here is that the perturbation series converges beyond this point. If the power of $(\alpha - \beta)$ appearing in the higher order terms of the perturbation series had increased, the convergence of the series under condition (2.51) would fail, but under a situation of properly bounded initial functions, equation (2.18) shows that only the first power of $(\alpha - \beta)$ enters for any perturbation function and the convergence persists.

<u>Geometry of the shock development region</u>

The correct introduction of a shock into the region of multiple-valued solutions requires some exploration of the geometry of the mapping of the (α, β)-plane into the (x, t)-plane in the region. In particular the earliest point in x and t at

which the envelope appears is of interest. Such an extremum for x is given by

$$dx = \frac{\partial x}{\partial \alpha} d\alpha + \frac{\partial x}{\partial \beta} d\beta = 0, \tag{2.52}$$

or along the envelope defined by (2.45)

$$\frac{dx}{d\beta} = \frac{\partial x}{\partial \alpha}\frac{d\alpha}{d\beta} = 0. \tag{2.53}$$

Since from the assumption that a shock does not form in the other direction,

$$\frac{\partial x}{\partial \alpha} \neq 0, \tag{2.54}$$

the extremum is given by

$$\frac{d\alpha}{d\beta} = 0. \tag{2.56}$$

Finally, since from the equations (2.1) for $(u - c) \neq 0$

$$\frac{\partial x}{\partial \beta} = 0 \implies \frac{\partial t}{\partial \beta} = 0,$$

the extremum for both x and t is given by (2.56).

For simplicity, the variables of the problem are now shifted to make the extremum of the envelope occur at $x = t = 0$ in the (x,t)-plane and at $\alpha = \beta = 0$ in the (α,β)-plane. An expansion of the functions x, t, u, c in terms of α and β near this extremum is then made in the form

$$\begin{aligned}
x &= x_{10}\alpha + x_{11}\alpha\beta + x_{20}\alpha^2 + x_{02}\beta^2 + x_{12}\alpha\beta^2 + x_{21}\alpha^2\beta + x_{30}\alpha^3 + x_{03}\beta^3 + \ldots \\
t &= t_{10}\alpha + t_{11}\alpha\beta + \ \ldots \\
u &= \varepsilon(u_{00} + u_{10}\alpha + u_{01}\beta + \ldots) \\
c &= c_0 + \varepsilon(c_{00} + c_{10}\alpha + c_{01}\beta + \ldots)
\end{aligned} \tag{2.57}$$

where x_{10}, etc., are constants, and where the fact $\frac{\partial x}{\partial \beta} = \frac{\partial t}{\partial \beta} = 0$ near the extremum, $\alpha = \beta = 0$, has been used.

Also it may be shown that at the extremum,

$$x_{\beta\beta} = t_{\beta\beta} = 0 \tag{2.58}$$

so that the envelope is cusped..

The proof of (2.58) follows from (2.56) which shows that the image of the envelope in the (α,β)-plane has an expansion of the form

$$\alpha = k\beta^2 + 0(\beta^3) + \dots, \; k = \text{constant} \tag{2.59}$$

Then from (2.57)

$$x_\beta = x_{11}\alpha + 2x_{02}\beta + 2x_{12}\alpha\beta + 3x_{03}\beta^2 + \dots ,$$

or along (2.59)

$$x_\beta = x_{11}k\beta^2 + 2x_{02}\beta + 2x_{12}k\beta^3 + 3x_{03}\beta^2 + 0(\beta^3) \; . \tag{2.60}$$

If (2.60) is to represent x_β correctly for small finite distances in β along the locus, $x_\beta = 0$, then the coefficient of β must vanish,

$$x_{02} = 0, \tag{2.61}$$

and the terms in β^2, β^3 etc. must balance, e.g.

$$x_{11}k = -3x_{03}. \tag{2.62}$$

Relations of the type (2.62) are demonstrated in a later example where the series (2.57) are given explicitly, (page49).

From (2.61) and similar considerations for t, the equation (2.58) is proved and the envelope is cusped. The series (2.57) become

$$x = x_{10}\alpha + x_{11}\alpha\beta + x_{20}\alpha^2 + x_{12}\alpha\beta^2 + x_{21}\alpha^2\beta + x_{03}\beta^3 + x_{30}\alpha^3 + \ldots$$

$$t = t_{10}\alpha + t_{11}\alpha\beta + t_{20}\alpha^2 + t_{12}\alpha\beta^2 + \ldots$$

$$u = \varepsilon\left[u_{00} + u_{10}\alpha + u_{01}\beta + \ldots\right] \qquad (2.57')$$

$$c = c_0 + \varepsilon\left[c_{00} + c_{10}\alpha + c_{01}\beta + \ldots\right] \quad .$$

For reference in the discussion sketches of the mapping near the cusp are shown in Fig. 1. In 1A is shown the image of the envelope, (2.59), in the (α,β)-plane. 1B shows the envelope in the (x,t)-plane and includes sketches of the triple fold. 1C and 1D plot $x(\beta)$ and $t(\beta)$ for constant α, and show the increase in the area of the folded region as α increases from its value at the cusp.

<u>Location of the shock wave</u>

The triple mapping leads one to examine the possibility of introducing a shock front which will allow the jumps in velocity and density between the outer sheets of the fold and which will allow therefore a continuation of the solution through the shock. These remarks apply of course only to regions of weak shock since the entropy change through the shock, which we are tacitly neglecting, affects quantities of the third order in the shock strength.

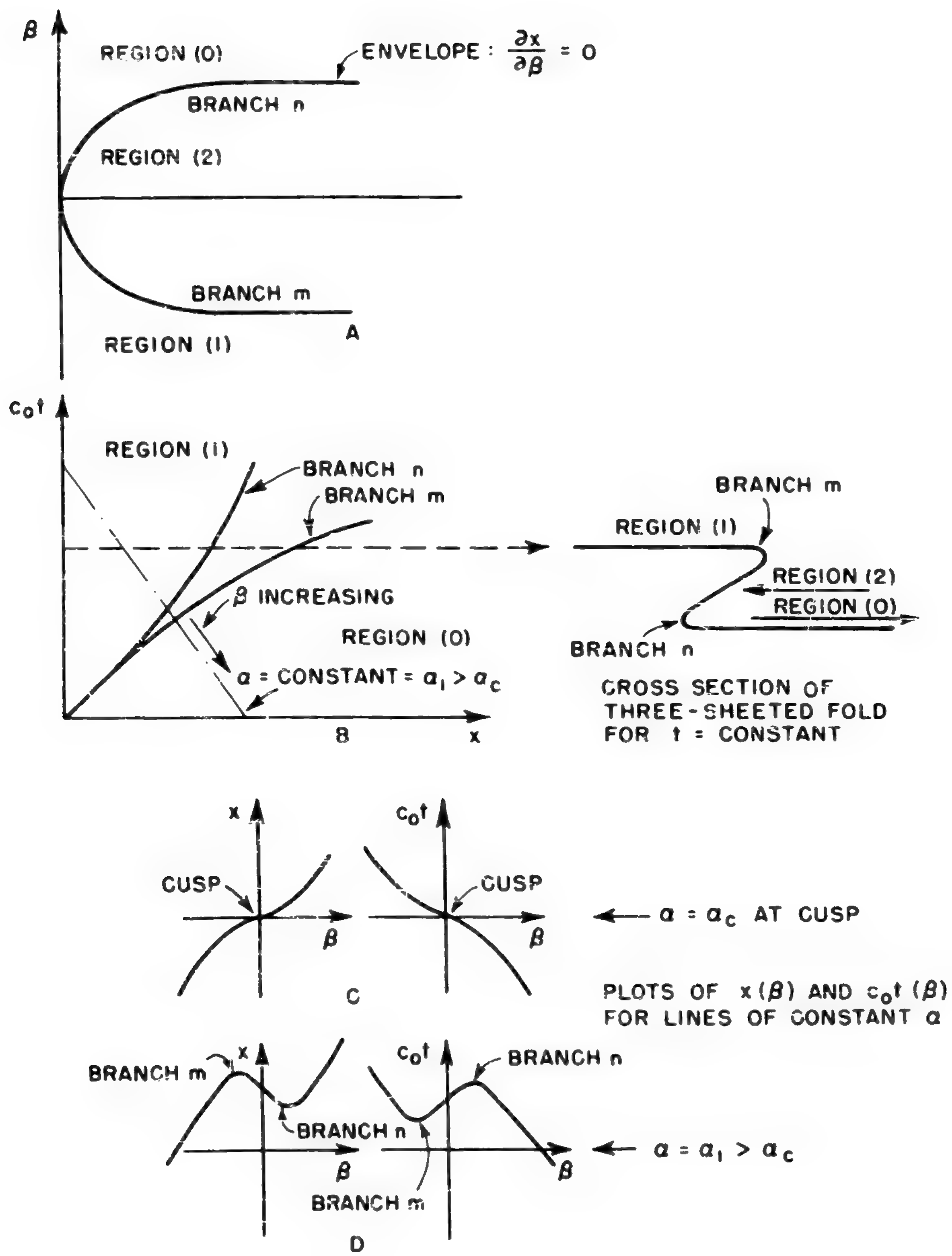


FIG. I LOCUS OF THE ENVELOPE: $\frac{\partial x}{\partial \beta}(\alpha_1 \beta) = 0$

The shock is to be located (see Fig. 1A and B) in the (x,t)-plane on the upper sheet (region (1)) just to the left of branch m, corresponding to a location on the lower sheet (region (0)) just to the right of branch n. Thus in the (α,β)-plane the image of the shock falls outside the region (2) and behaves near the origin like

$$\alpha = \bar{k}\beta^2 + \ldots, \text{ where } \bar{k} > k \text{ of equation (2.59).} \tag{2.63}$$

Then for a point (x,t) on the shock on the upper sheet, there must correspond the same point (x,t) on the lower sheet, and the jumps on velocity and density between the two sheets must obey the shock transition equations (Rankine-Hugoniot conditions [4]).

To this end let the shock be represented parametrically by a parameter σ, representing distance along the shock such that, in view of (2.59)

$$\left.\begin{aligned} \alpha &= c_2\sigma^2 + c_3\sigma^3 + \ldots \\ \beta &= d_1\sigma + d_2\sigma^2 + d_3\sigma^3 + \ldots \end{aligned}\right\} \text{ on the upper sheet, and} \tag{2.64a}$$

$$\left.\begin{aligned} \alpha &= e_2\sigma^2 + e_3\sigma^3 + \ldots \\ \beta &= f_1\sigma + f_2\sigma^2 + f_3\sigma^3 + \cdots \end{aligned}\right\} \text{ on the lower sheet,} \tag{2.64b}$$

where c_2, d_1, ... are constants.

In order that points (x,t) on the outer sheets may correspond for a given σ, equations (2.57') show

$$x_{10}(c_2\sigma^2 + c_3\sigma^3) + x_{11}(c_2 d_1\sigma^3) + x_{03}d_1^3\sigma^3 + O(\sigma^4) = x_{10}(e_2\sigma^2 + e_3\sigma^3) +$$

$$x_{11}(e_2 f_1\sigma^3) + x_{03}f_1^3\sigma^3 + O(\sigma^4) \quad ,$$

or, from the σ^2 terms,

$$c_2 = e_2. \tag{2.65}$$

The shock transition equations are conservation of mass,

$$\rho_o v_o = \rho_1 v_1 = m, \qquad \begin{aligned} v_1 &= u_1 - U \\ U &= \text{shock velocity} \end{aligned} \tag{2.66a}$$

conservation of momentum:

$$\rho_o v_o^2 + p_o = \rho_1 v_1^2 + p_1 \tag{2.66b}$$

conservation of energy,

$$\frac{v_o^2}{2} + e_o + \frac{p_o}{\rho_o} = \frac{v_1^2}{2} + e_1 + \frac{p_1}{\rho_1} \tag{2.66c}$$

Increase of entropy

$$mS_o \leq mS_1 \quad ; \quad S = \text{entropy.} \tag{2.66d}$$

In the solution being considered entropy change is to be neglected which means that the last equation above is irrelevant, and also that the energy equation (2.66c) reduces to Bernoulli's law and is already satisfied by a solution of the original equations (§55,[4]). To satisfy the other equations use is made of equations (2.57'), (2.66), and the further expansion

$$\rho = \rho_0 + \varepsilon(\rho_{00} + \rho_{10}\alpha + \rho_{01}\beta + \rho_{02}\beta^2 + \ldots) \quad (2.67)$$

and (2.66a) is evaluated on the upper and lower sheets along the shock.

Taking the shock velocity as

$$U = U^{(0)} + \varepsilon U^{(1)} + \ldots \quad (2.68)$$

and substituting for the quantities in (2.66a) evaluated on the two sheets of (2.64, a, b) gives as the conditions obtained by setting the coefficients of σ and σ^2 equal to zero respectively,

$$(d_1 - f_1)\left(-\rho_{01}U^{(0)} + (\rho_0 + \varepsilon\rho_{00})u_{01} + \varepsilon\rho_{01}u_{00}\right) = 0$$

$$-(d_1 - f_1)\varepsilon\rho_{01}U^{(1)} + (d_1^2 - f_1^2)\left[(\varepsilon\rho_0 + \varepsilon^2\rho_{0c})u_{02} + \varepsilon^2\rho_{01}u_{01} - U^{(0)}\varepsilon\rho_{02}\right] = 0 \quad (2.69,a,b)$$

In these expressions, the relation $d_1 = f_1$ is not allowed, since in equations (2.64, a, b) β must have different signs on the two outer sheets of the shock (see Fig. 1A). Thus one obtains from (2.69a)

$$U^{(0)} = u_{00} + \left(\frac{\rho_0 + \varepsilon\rho_{00}}{\rho_{01}}\right)u_{01}$$

or since,

$$u_{01} = \frac{2}{\gamma-1}c_{01} \text{ and } \frac{\rho_0 + \varepsilon\rho_{00}}{\rho_{01}} = \frac{\gamma-1}{2}\frac{(c_0 + \varepsilon c_{00})}{c_{01}}, \quad (2.70)$$

$$U^{(0)} = c_0 + \varepsilon(c_{00} + u_{00}) \quad (2.71)$$

Then one may show from (2.69b) that

$$\begin{aligned} d_1 &= -f_1 \\ U^{(1)} &= 0 \end{aligned} \tag{2.72}$$

is a satisfactory solution since to this order the required relation on the shock,

$$\frac{dx/ds}{dt/ds} = U \qquad \text{is satisfied.} \tag{2.73}$$

The equations for the shock become

$$\left.\begin{aligned} \alpha &= c_2\sigma^2 + \ldots \\ \beta &= d_1\sigma + \ldots \end{aligned}\right\} \text{ on the upper sheet}$$

$$\left.\begin{aligned} \alpha &= c_2\sigma^2 + \ldots \\ \beta &= -d_1\sigma + \ldots \end{aligned}\right\} \text{ on the lower sheet.} \tag{2.74a,b}$$

Furthermore the speed of the shock, that is its slope in the (x,t)-plane is the average of the slopes of the β -characteristics forming the two branches of the envelope. This is shown by using equations (2.1) and (2.57') to obtain

$$\left.\frac{dx}{dt}\right)_{\beta=\text{constant}} = \frac{x_\alpha}{t_\alpha} = u + c = c_o + \varepsilon\Big[(u_{00}+c_{00}) + (u_{01}+c_{01})\beta + (u_{10}+c_{10})\alpha + (u_{02}+c_{02})\beta^2 + \ldots\Big]. \tag{2.75}$$

Thus for equal distances in β ($+\bar{\beta}$ and $-\bar{\beta}$) along the two branches of the envelope,

$$\frac{1}{2}\left[\left(\frac{dk}{dt}\right)_{+\bar{\beta}} + \left(\frac{dx}{dt}\right)_{-\bar{\beta}}\right] = \frac{1}{2}\left\{2c_0 + 2\varepsilon\left[(u_{00} + c_{00}) + (\bar{\rho}-\bar{\beta})(u_{01} + c_{01}) + O(\bar{\beta}^2)\right]\right\}$$

$$= c_0 + \varepsilon(u_{00} + c_{00}) \doteq U. \qquad (2.76)$$

The momentum equation (2.66b) holds through order $\varepsilon\sigma^2$ as one may show most easily by using the equivalent relation

$$\frac{p_1 - p_0}{\rho_1 - \rho_0} = \frac{\gamma(p_1 + p_0)}{\rho_1 + \rho_0} \quad ,$$

and expressing p and ρ on the two sides of the shock in terms of an $(\alpha(\sigma), \beta(\sigma))$ expansion.

It is found then that if the shock wave is properly imposed on the region of triple mapping, the shock transition equations (2.66) may be used to account for the jumps in the physical quantities. The equations hold through order $\varepsilon\sigma^2$ where $\varepsilon\sigma$ is being used as a measure of shock strength since for example from equation (2.67)

$$\rho = \rho_0 + \varepsilon(\rho_{00} + \rho_{01}\beta + \rho_{10}\alpha + \ldots),$$

which along the shock becomes

$$\rho = \rho_0 + \varepsilon\left[\rho_{00} + \rho_{01}\,(\pm d_1\sigma) + O(\sigma^2)\right]$$

shows that a jump in density is of the order of a jump in $\varepsilon\sigma$.

As far as the error due to neglect of entropy change is concerned, one would estimate it as the cube of the shock strength, or $(\varepsilon\sigma)^3$, but actually the error involved is larger

since the shock transition equations are satisfied only through order $\varepsilon\sigma^2$ and involve an error of order $\varepsilon\sigma^3$. To find the actual error incurred by pushing the solution a distance x beyond the point of shock formation, equations (2.57') and (2.74a,b) are used to show that since

$$x \sim \beta^2 \sim \sigma^2,$$

an error of order $\varepsilon\sigma^3$ corresponds to an

$$\text{error} \sim \varepsilon x^{3/2} \tag{2.77}$$

in traveling a distance x along the shock.

<u>Illustrative example</u>

The example used in the first section is reexamined here from the standpoint of shock development. The periodic initial distribution of the problem shows that not one, but an infinite set of shocks will develop in a repeating pattern from the infinite initial set of compression waves. Furthermore envelopes will occur for both the α - and β -sets of characteristics and a sketch of the pattern of envelopes is given on page 51. However, since this study does not purport to study interactions, a single developing shock wave will be isolated and used to illustrate the points made earlier in this section.

From equations (2.34), the envelope, $\frac{\partial x}{\partial \beta} = 0$, is found to be

$$\alpha - \beta = \frac{1}{\varepsilon \bar{A} \sin\beta} + \frac{\cos\alpha + \frac{A+B}{A}\cos\beta}{\sin\beta} + \frac{\varepsilon R_1}{\sin\beta} + O(\varepsilon^2) \tag{2.78}$$

where

$$R_1 = (2\bar{A}+4\bar{B})\cos\beta\cos\alpha - 3\bar{B}\sin\alpha\sin\beta - (\frac{\bar{A}}{2}+\frac{\bar{B}}{4})\cos 2\alpha +$$
$$(-\frac{\bar{A}}{2} - \frac{13}{4}\bar{B})\cos 2\beta + (\bar{A}+\frac{\bar{B}}{2}) + \frac{(2\bar{A}+\bar{B})^2}{2\bar{A}}\sin\beta(1+\cos 2\beta).$$

If the problem is made more definite by choosing

$$\gamma = 1.4$$
$$\bar{A} = 3/2$$
$$\bar{B} = 1$$
$$\varepsilon = 1/100$$

then the envelope becomes

$$\alpha = \beta + \frac{200}{3\sin\beta} + \frac{\cos\alpha + 5/3\cos\beta}{\sin\beta} + \frac{\varepsilon R_1}{\sin\beta} + O(\varepsilon^2) \qquad (2.79)$$

where

$$R_1 = 7\cos\beta\cos\alpha - 3\sin\alpha\sin\beta - \cos 2\alpha - 4\cos 2\beta + 2 + \frac{16}{3}\sin\beta(1+\cos 2\beta).$$

The cusps of the envelope are located at $\frac{d\alpha}{d\beta} = 0$ and choosing the the branch which has a cusp near $\beta = \pi/2$, the cusp is at

$$\alpha_c = 69.2$$
$$\beta_c = 1.58$$
$$x_c = 34.9$$
$$c_o t_c = 33.3$$

Then if the problem is shifted to put the cusp at the origin in both the (α,β)- and (x,t)-planes, the following expansions of the type (2.59) and (2.57') apply.

The equation of the envelope in the (α,β)-plane near the cusp has the expansion,

$$\alpha = 31.9(\beta)^2 + \dots . \qquad (2.80)$$

The quantities x, $c_o t$, u, and c near the cusp exhibit the following behavior.

$$\begin{aligned} x &\simeq +.51\alpha - .008\alpha\beta + .08\beta^3 + \dots \\ c_o t &\simeq +.52\alpha + .008\alpha\beta - .08\beta^3 + \dots \\ \frac{u}{c_o} &\simeq \frac{\varepsilon}{\gamma-1}\left[-1.008 - \beta + .06\alpha + .005\beta^2 + .499\alpha^2 + \dots\right] \\ \frac{c}{c_o} &\simeq 1 + \frac{\varepsilon}{2}\left[0.988 - \beta - .06\alpha + .005\beta^2 - .499\alpha^2 + \dots\right]. \end{aligned} \qquad (2.81)$$

That is, the expressions (2.81) do assume the form predicted in (2.57'), and in fact relations of the type (2.62) are verified since in

$$x_{11}k = -3x_{03}$$

The values are

$$k = 31.9$$

$$x_{11} = -.008$$

$$x_{03} = +.08$$

and the equality holds to within the accuracy of the calculation.

The error involved in this particular problem may be estimated again from the failure of the shock transition equations to hold through order $\varepsilon\sigma^3$. From the first of equations (2.81)

$$x \simeq 16\beta^2 + O(\beta^3) \simeq 16\sigma^2 + O(\sigma^3)$$

so that $\quad \varepsilon\sigma^3 \simeq \varepsilon\left(\frac{x}{16}\right)^{3/2} = \frac{\varepsilon}{64}x^{3/2}, \qquad (2.82)$

and the solution may be continued for a distance x along the shock with an error given by (2.82).

As stated previously, the actual flow pattern in the (x,t)-plane is a configuration of intersecting compression and expansion waves, so that depending on the time of shock development the compression wave traveling to the right and steepening has encountered a certain number of leftward-traveling compression and rarefaction waves. From the fact that a shock developes at a time, t, of order $\frac{1}{\varepsilon}$ one sees that the patterns of interaction vary with ε, and that in fact for certain epsilons the shocks traveling in different directions would tend to form at the same time, that is, the envelopes,

$$\frac{\partial x}{\partial \beta} = 0 \quad : \quad \alpha = \beta + \frac{1}{\varepsilon \bar{A} \sin\beta} + \frac{\cos\alpha + \frac{\bar{A}+\bar{B}}{\bar{A}}\cos\beta}{\sin\beta} + \frac{\varepsilon R_1}{\sin\beta} + O(\varepsilon^2) \qquad (2.83)$$

$$\frac{\partial x}{\partial \alpha} = 0 \quad : \quad \beta = \alpha + \frac{1}{\varepsilon \bar{A} \sin\alpha} + \frac{\cos\beta + \frac{\bar{A}+\bar{B}}{\bar{A}}\cos\alpha}{\sin\alpha} + \frac{\varepsilon R_2}{\sin\alpha} + O(\varepsilon^2) \qquad (2.84)$$

where $R_1(\alpha,\beta) = R_2(\beta,\alpha)$

would each have a cusp at the same values of α and β. In Fig. 2 on page 51 are shown sketches of the pattern in the (x,t)-plane and of the effect of ε on the pattern of interaction. The initial distribution of c is superimposed on the diagrams. For the problem discussed above with $\varepsilon = 1/100$, the shock development is delayed so long that many interactions

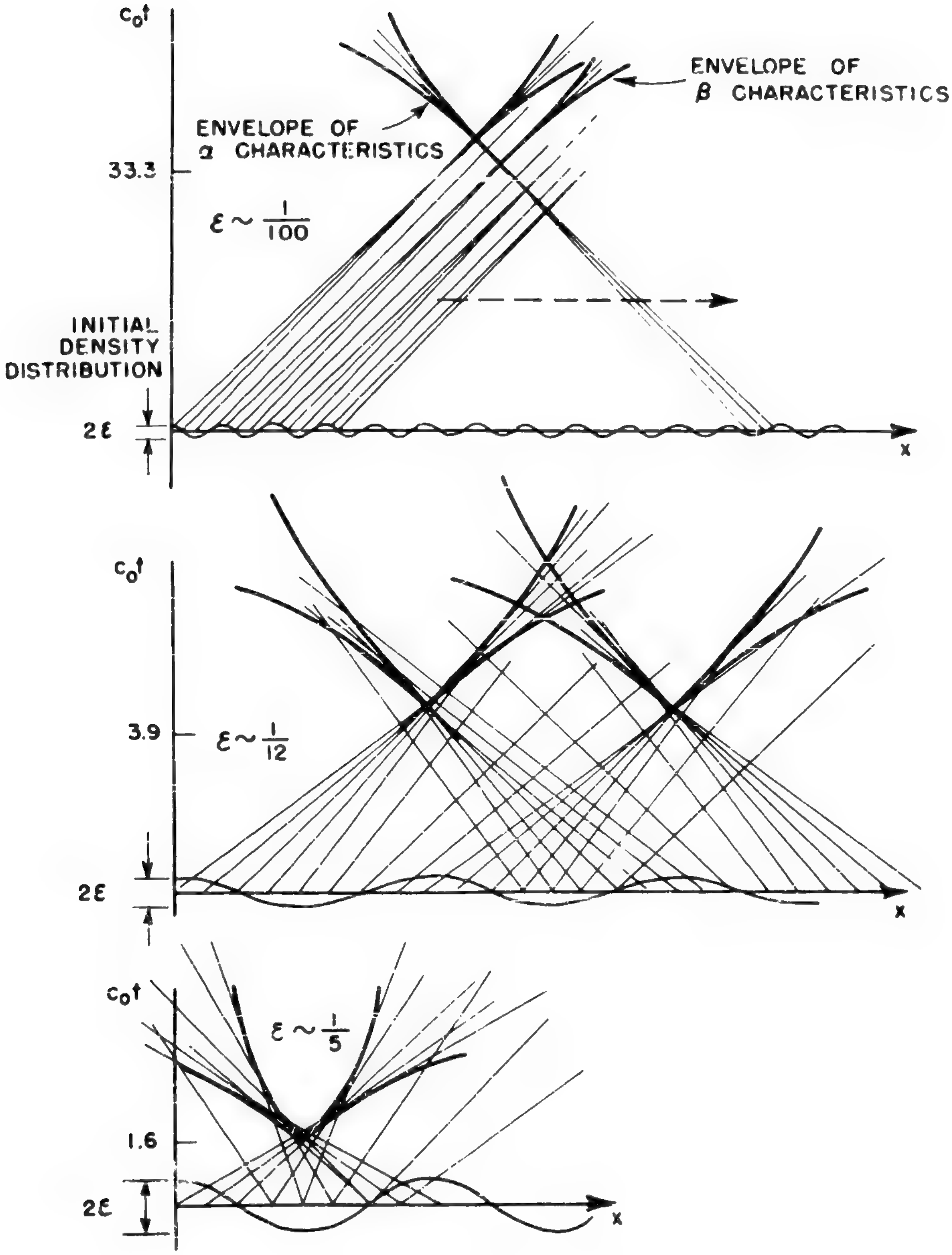


FIG. 2 PATTERNS OF SHOCK FORMATION FOR DIFFERENT VALUES OF ε

have occurred; for $\varepsilon \sim \frac{1}{12}$ on the other hand only one interaction has occurred, and for some $\varepsilon > 1/6$ the pattern of Fig. 2c occurs. This last estimate cannot be made with assurance from the equations, since the ε is larger than that allowed on page 28 from convergence considerations, but obviously there will be a first ε to give such a region, and there will also be a set of increasing ε's involving such regions. The behavior of the solution in these neighborhoods has not been explored in detail and might prove of considerable interest.

B. Cylindrical and Spherical Wave Propagation

Introduction

The success with which the coordinate-perturbation type of solution was able to describe the region of shock formation for the case of the plane wave, leads one to hope that this type of solution might be extended to describe shock regions for the spherical and axi-symmetrical cases. However, in the plane wave case the right to carry the solution up to and through the shock depended on a definite proof of the convergence of the series, and unfortunately such a convergence proof has not yet been found for the higher dimension cases. Thus the right to extend the solution for these cases must be held in abeyance.

Nevertheless it seems probable if not yet justified that the extension can be made, and in fact G. B. Whitham [23], [24]

by allowing a perturbation to the direction of one set of characteristics has had considerable success in studying the regions of shocks.

Some aspects of the matter will be considered here. In section 1 the general form assumed by the equations for the sequence of perturbation functions will be given together with a brief discussion of the type of convergence conjectured for the series solutions.

Section 2 compares the results of section 1 with those found by Whitham in [23].

<u>Section 1</u>: Perturbation solutions

In terms of characteristic variables α and β, the equations describing spherical or cylindrical wave motion become (§ 23, [4])

$$\begin{aligned} x_\alpha &= (u + c)t_\alpha & \left(\frac{\gamma-1}{2}\right)\left[u_\alpha + \frac{(n-1)uc}{x}t_\alpha\right] + c_\alpha &= 0 \\ x_\beta &= (u - c)t_\beta & \left(\frac{\gamma-1}{2}\right)\left[u_\beta - \frac{(n-1)uc}{x}t_\beta\right] - c_\beta &= 0, \end{aligned} \tag{2.85}$$

where n is the number of dimensions involved, and where x and t are the space and time coordinates respectively, and u and c the velocity and speed of sound.

For simplicity the initial conditions will be given for a gas initially at rest, and as before the initial line will be defined by setting both α and β equal to x.

Then on $\alpha = \beta$,

$$\begin{aligned} x &= \alpha = \beta \\ t &= 0 \\ u &= 0 \\ c &= c_0 + \epsilon\,\bar{c}^{(1)}(x). \end{aligned} \tag{2.86}$$

If a perturbation solution is sought in the form

$$\begin{aligned} x &= x^{(0)} + \varepsilon x^{(1)} + \varepsilon^2 x^{(2)} + \dots \\ t &= t^{(0)} + \varepsilon t^{(1)} + \varepsilon^2 t^{(2)} + \dots \\ u &= u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \dots \\ c &= c^{(0)} + \varepsilon c^{(1)} + \varepsilon^2 c^{(2)} + \dots \end{aligned} \tag{2.87}$$

from (2.85) and (2.86) the terms free from ε are found to be

$$\begin{aligned} x^{(0)} &= \frac{\alpha+\beta}{2} \\ c_o t^{(0)} &= \frac{\alpha-\beta}{2} \\ u^{(0)} &= 0 \\ c^{(0)} &= c_0. \end{aligned} \tag{2.88}$$

Using (2.88) the equations for the first order perturbation functions $u^{(1)}$ and $c^{(1)}$ become

$$\begin{aligned} \frac{\gamma-1}{2}\left[u_\alpha^{(1)} + \frac{(n-1)}{\alpha+\beta}u^{(1)}\right] + c_\alpha^{(1)} &= 0 \\ \frac{\gamma-1}{2}\left[u_\beta^{(1)} + \frac{(n-1)}{\alpha+\beta}u^{(1)}\right] - c_\beta^{(1)} &= 0, \end{aligned} \tag{2.89}$$

or, eliminating $c^{(1)}$,

$$u_{\alpha\beta}^{(1)} + \frac{(n-1)\left[u_\alpha^{(1)} + u_\beta^{(1)}\right]}{\alpha+\beta} - \frac{(n-1)u^{(1)}}{(\alpha+\beta)^2} = 0. \tag{2.90}$$

At this stage, both to expedite the calculations, and to obtain a solution in a form comparable with Whitham's work on spherical blast, the problem will be specialized to the spherical case,

$$n = 3.$$

Then (2.90) becomes

$$u^{(1)}_{\alpha\beta} + \frac{\left[u^{(1)}_{\alpha} + u^{(1)}_{\beta}\right]}{\alpha+\beta} - \frac{2u^{(1)}}{(\alpha+\beta)^2} = 0 \tag{2.91}$$

which may be shown to have the solution

$$u^{(1)} = \frac{\frac{df(\alpha)}{d\alpha} + \frac{dg(\beta)}{d\beta}}{\alpha+\beta} - \frac{2\left[f(\alpha) + g(\beta)\right]}{(\alpha+\beta)2},$$

where f and g are arbitrary functions determined from the initial conditions. Finally using the initial conditions and the equations, a complete set of solutions for the first order perturbations is,

$$\begin{aligned}
u^{(1)} &= \frac{1}{\alpha+\beta}\left[\frac{df(\alpha)}{d\alpha} - \frac{df(\beta)}{d\beta}\right] - \frac{2}{(\alpha+\beta)^2}\left[f(\alpha) - f(\beta)\right] \\
c^{(1)} &= -\frac{(\gamma-1)}{2}\cdot\frac{1}{\alpha+\beta}\left[\frac{df(\beta)}{d\beta} + \frac{df(\alpha)}{d\alpha}\right] \\
x^{(1)} &= -\frac{(\gamma-1)}{4c_0}\Bigg\{\frac{f(\alpha)+f(\beta)}{\alpha+\beta} + \ln(\alpha+\beta)\left[\frac{df(\beta)}{d\beta} + \frac{df(\alpha)}{d\alpha}\right] + \int_{\beta}^{\alpha}\left[\frac{1}{(\mu+\beta)^2} + \frac{1}{(\mu+\alpha)^2}\right]f(\mu)\,d\mu \\
&\qquad - \frac{f(\beta)}{2\beta} - \frac{f(\alpha)}{2\alpha} - \frac{df(\beta)}{d\beta}\ln 2\beta - \frac{df(\alpha)}{d\alpha}\ln 2\alpha\Bigg\} \\
t^{(1)} &= \frac{(\gamma+1)}{8c_0^2}\Bigg\{\left(\frac{12-\gamma}{\gamma+1}\right)\frac{f(\alpha)-f(\beta)}{\alpha+\beta} + \ln(\alpha+\beta)\left[\frac{df(\beta)}{d\beta} - \frac{df(\alpha)}{d\alpha}\right] - \int_{\beta}^{\alpha}\left[\frac{1}{(\mu+\beta)^2} - \frac{1}{(\mu+\alpha)^2}\right]f(\mu)\,d\mu - \\
&\qquad - \frac{f(\beta)}{2\beta} + \frac{f(\alpha)}{2\alpha} - \frac{df(\beta)}{d\beta}\ln 2\beta + \frac{df(\alpha)}{d\alpha}\ln 2\alpha\Bigg\}
\end{aligned} \tag{2.92}$$

where $\frac{df(y)}{dy} = \frac{-2y}{\gamma-1}\bar{c}^{(1)}(y)$, so that the equation for $c^{(1)}$ becomes $c^{(1)} = \left[\alpha c^{(1)}(\alpha) + \beta c^{(1)}(\beta)\right]/(\alpha+\beta)$. (2.93)

However, at the next step, when the higher order perturbation functions, $u^{(2)}$ etc., are to be found, the term

$\frac{(n-1)uct_\alpha}{x}$ in (2.85) introduces non-homogeneous terms into the equations and makes the solutions of the equation much more difficult to obtain. It is exactly this difficulty in finding the expressions for the solutions which has deterred the proof of the convergence of the perturbation series.

At any rate the form of the equations for the higher order perturbations can be given for a general n^{th} order perturbation. In view of the fact that

$$\frac{1}{x} = \frac{1}{x^{(0)}}\left\{1 - \frac{\varepsilon x^{(1)}}{x^{(0)}} + \varepsilon^2\left(-\frac{x^{(2)}}{x^{(0)}} + \left(\frac{x^{(1)}}{x^{(0)}}\right)^2\right) + \cdots + \varepsilon^j \sum_{\substack{m\mu + l\lambda + \cdots + z\zeta = j \\ m, l, z, \cdots > 0 \\ m \neq l \neq \cdots \neq z}} \frac{(x^{(m)})^\mu (x^{(l)})^\lambda \cdots (x^{(z)})^\zeta}{(-x^{(0)})^{\mu+\lambda+\cdots+\zeta}} + \cdots\right\}$$

and with $x^{(0)} = \frac{\alpha+\beta}{2}$, the general equations (2.85) become,

$$\frac{\gamma-1}{2}\left[u_\alpha^{(n)} + \frac{2}{\alpha+\beta}u^{(n)}\right] + c_\alpha^{(n)} = -\frac{2(\gamma-1)}{\alpha+\beta}\sum_{\substack{p+q+r+j=n \\ p,q,r,j<n}} u^{(p)}c^{(q)}t_\alpha^{(r)} \text{ times } \cdot \sum \frac{(x^{(m)})^\mu (x^{(l)})^\lambda \cdots (x^{(z)})^\zeta}{\left(-\frac{\alpha+\beta}{2}\right)^{\mu+\lambda+\cdots+\zeta}},$$

$$\frac{\gamma-1}{2}\left[u_\beta^{(n)} + \frac{2}{\alpha+\beta}u^{(n)}\right] - c_\beta^{(u)} = +\frac{2(\gamma-1)}{\alpha+\beta}\sum_{\substack{p+q+r+j=n \\ p,q,r,j<n}} u^{(p)}c^{(q)}t_\beta^{(r)} \text{ times } \cdot \sum \frac{(x^{(m)})^\mu (x^{(l)})^\lambda \cdots (x^{(z)})^\zeta}{\left(-\frac{\alpha+\beta}{2}\right)^{\mu+\lambda+\cdots+\zeta}}, \qquad (2.94)$$

$$x_\alpha^{(n)} - c_o t_\alpha^{(n)} = \sum_{j=1}^{n} \left[u^{(j)} + c^{(j)}\right] t_\alpha^{(n-j)}$$

$$x_\beta^{(n)} + c_o t_\beta^{(n)} = \sum_{j=1}^{n} \left[u^{(j)} - c^{(j)}\right] t_\beta^{(n-j)} \quad .$$

Now even though a proof of the convergence of the perturbation series has not been found, it seems worthwhile to discuss the region where a shock might be expected to develop, and to attempt an estimate of the size of the terms in the perturbation series in such a vicinity. To this end, as in the plane wave case, it will be assumed that only one set of characteristics folds into a triple mapping at one time. The region will be considered asymptotically in terms of only one characteristic variable.

First of all if the perturbation, $\tilde{c}^{(1)}$, to the quantity, c, is to vanish initially for large distances, (2.93) shows that asymptotically

$$\frac{df(y)}{dy} \sim \text{constant.} \tag{2.95}$$

Then from the equation for $x^{(1)}$ in (2.92), under the assumption

$$\beta >> \alpha \tag{2.96}$$

one finds $x^{(1)} \sim \ln \beta$, $c_o t^{(1)} \sim \ln \beta$.

Considering therefore the asymptotic form for the expression

$$x = x^{(0)} + \varepsilon x^{(1)} + \cdots \sim O(\beta) + O(\varepsilon \ln \beta) + \ldots, \tag{2.97}$$

We note that for the deviation from the linear case to be great enough to allow the characteristics to cross, the second term must be of order 1, or

$$\varepsilon = O\left(\frac{1}{\ln \beta}\right) \tag{2.98}$$

(c.f. a similar conclusion by Whitham, §4[23]). The question then arises as to the order of the higher perturbation terms in this vicinity; that is, does the series converge in such a region? In answer only the unproved conjecture can be stated here that asymptotically

$$\begin{aligned} x^{(n)} &\sim c_o t^{(n)} \sim \left(\frac{\ln \beta}{\beta}\right)^{n-1} \\ u^{(n)} &\sim c^{(n)} \sim \left(\frac{\ln \beta}{\beta^n}\right)^{n-1}, \quad n > 1. \end{aligned} \tag{2.99}$$

The heuristic argument leading to the relations (2.99) is based on a consideration of the non-homogeneous terms in (2.94). For example the equation for the second order perturbation has terms of the form,

$$u_\beta^{(2)} + \ \ldots = \frac{4}{(\alpha+\beta)^2}\, x^{(1)} u^{(1)} + \ \ldots.,$$

and if asymptotically

$$x^{(1)} \sim \ln \beta$$

$$u^{(1)} \sim \frac{1}{\beta},$$

then asymptotically $\quad u_\beta^{(2)} + \ \ldots \sim \dfrac{\ln \beta}{\beta^3}\ ,$

and then from this without being able to justify the step, we

assume

$$u^{(2)} \sim \frac{\ln \beta}{\beta^2}. \qquad (2.100)$$

Relying on the conjectured relations (2.99), the perturbation series would have the asymptotic orders,

$$x \sim O(\beta) + O(\varepsilon \ln \beta) + O(\varepsilon^2 \frac{\ln \beta}{\beta}) + O(\varepsilon^3 \frac{\ln^2 \beta}{\beta^2}) + \ldots, \qquad (2.101)$$

or in the vicinity $\ln \beta \sim \frac{1}{\varepsilon}$, indicative of possible shock development,

$$x \sim O(\beta) + O(1) + O(\frac{\varepsilon}{\beta}) + O(\frac{\varepsilon}{\beta^2}) + \ldots, \qquad (2.102)$$

giving a series still convergent in a perturbation sense.

The work done by Whitham contains perturbation series whose behavior at large distances agrees with that found in this section, and is discussed in the following section.

Section 2: Comparison with Whitham's work

Whitham, in his paper on "The Propagation of Spherical Blast," [23], makes the point that in a strictly linearized wave propagation problem the characteristics are straight and parallel, and hence cannot cross as they would in actuality do near regions of shock development. To correct the situation he proposes that the solutions for velocity and density be considered as functions not of the linearized characteristic, $c_o t - x$, but of a true characteristic variable, z, where z is determined to be truly constant along a characteristic. The linear characteristic is to be corrected by the expansion,

$$c_0 t = x - z\log x - h(z) - (m_1(z)\log x + m_2(z))x^{-1} + \dots,$$

equation (9), [23], and this expansion together with expansions of the type,

$$\text{velocity, } u = -c_0 f(z)x^{-1} + c_0\Big[(b_1(z)\log x + b_2(z))x^{-2} + (c_1(z)\log x + c_2(z))x^{-3} + \dots\Big],$$

are to be used to find the solution of the problem.

In other words, a perturbation to the direction of only one set of characteristics is used and it is assumed that there is no significant nonlinearity in the other set. Such an approach differs of course from that we have used earlier in this report where perturbation series were considered for both variables x and t in terms of both sets of characteristics. In our case a correction is automatically effected for both characteristic directions, but for the spherical case the problem tends to become more complex than for the plane wave propagation.

The solutions which Whitham finds using the expansions above are

$$u = \frac{(c_0 K)(z)}{x} + \frac{c_0}{x^2}\left[-\frac{1}{2}Kz^2\log x - B_1\left(\log x + \frac{1}{2}\right) - K\int_0^z \zeta h'(\zeta)\,d\zeta - B_2 - \frac{1}{2}Kz^2 - K_1 z^2\right] + \dots,$$

equation (12), [23]

$$c_0 t = x - z\log x - h(z) - \frac{\left(\frac{1}{2}Kz^2 + B\right)\log x + K_2 z^2 + \frac{1}{4}(\gamma+5)B_1 + K\int_0^z \zeta h'(\zeta)\,d\zeta + B_2}{x} + \dots,$$

equation (13), [23]

where B_1 and B_2 are arbitrary constants, $k = \frac{2}{\gamma+1}$, $k_1 = k^2 - \frac{1}{4}k$, $k_2 = \frac{5}{4} + \frac{3}{2}k$, and where the function $h(x)$ is determined from initial conditions.

On the other hand, the solutions for these functions which we find using our kind of perturbation approach are equations of the type (2.92), which, before enforcing initial conditions, become

$$u^{(1)} = \frac{\frac{df(\alpha)}{d\alpha} + \frac{dg(\beta)}{d\beta}}{\alpha+\beta} - 2\frac{[f(\alpha) + g(\beta)]}{(\alpha+\beta)^2} \tag{2.103}$$

$$x^{(1)} - c_o t^{(1)} = \frac{\gamma+1}{4c_o}\frac{dg(\beta)}{d\beta}\log(\alpha+\beta) + \frac{1}{c_o(\alpha+\beta)}\left[g(\beta) + \frac{3-\gamma}{4}f(\alpha)\right] -$$

$$- \frac{(1+\gamma)}{4c_o}\int^{\alpha}\frac{f(\mu)\,d\mu}{(\mu+\beta)^2} + \tilde{x}_1(\beta) \tag{2.104}$$

where $\tilde{x}_1(\beta)$ is a function to be determined from the conditions of the problem.

Now in order to be able to compare Whitham's results with ours, several clarifications must be made. Primarily of course since his results are based on outgoing waves, our result must be simplified to contain functions of the characteristic β only. Further since we have results only through the first order, the comparison can be made only at this level. The equations to be compared become

$$u = \varepsilon\left[\frac{\frac{dg(\beta)}{d\beta}}{\alpha+\beta} - \frac{2g(\beta)}{(\alpha+\beta)^2}\right] + O(\varepsilon^2) \tag{2.105}$$

$$x - c_o t = \beta + \varepsilon\left[\frac{\gamma+1}{4c_o}\frac{dg(\beta)}{d\beta}\log(\alpha+\beta) + \frac{g(\beta)}{c_o(\alpha+\beta)} + \tilde{x}_1(\beta)\right] + O(\varepsilon^2) \tag{2.106}$$

Whitham does not expressly state that his solution is a perturbation one, but the fact that he considers a deviation of the characteristics from those given by the linear solution implies a perturbation technique. Thus from his equation (13), the quantity z which is constant along a characteristic also must act as a small order perturbation quantity (see also the discussion in section 4 of his paper) and so we expand z in a perturbation series

$$z = \varepsilon z^{(1)}(\beta) + \varepsilon^2 z^{(2)}(\beta) + \dots \quad (2.107)$$

From the same sort of considerations his terms of order one in both $(c_0 t - x)$ and u are set equal to zero. (Restriction of the problem to one of zero initial velocity does not alter the comparison.) From the terms of order one in his equations then

$$\begin{aligned} &B_1 \log x + \tfrac{1}{4}(\gamma + 5)B_1 + k\int_0^z \zeta h'(\zeta)\,d\zeta + B_2 = 0 \\ &B_1 \log x - \frac{B_1}{2} - k\int_0^z \zeta h'(\zeta)\,d\zeta - B_2 = 0, \end{aligned} \quad (2.108)$$

so that $B_1 = 0$ and B_2 must be such that

$$k\int_0^z \zeta h'(\zeta)\,d\zeta = -B_2 + \varepsilon \mathcal{J}^{(1)}(\beta) + \varepsilon^2 \mathcal{J}^{(2)}(\beta) + \dots \quad (2.109)$$

In fact it is true that B_1 becomes zero in the problems Witham considers and that (2.109) or an analogue thereof appears in his section 3 in the form

$$k \int \zeta h'(\zeta) d\zeta = k \left[\frac{b^2}{2} - h_1(z) + zh(z) \right]$$

$$B_2 = -1/2kb^2$$

Using equations (2.107), (2.109), and also the expansions

$$x = \frac{\alpha+\beta}{2} + \varepsilon x^{(1)}(\alpha,\beta) + O(\varepsilon^2)$$

$$h(z) = H^{(0)}(\beta) + \varepsilon H^{(1)}(\beta) + \varepsilon^2 H^2(\beta) + \ldots \qquad (2.110)$$

in his equations (12) and (13), gives:

$$u = \varepsilon \left[\frac{2c_0 k Z^{(1)}_{(\beta)}}{\alpha+\beta} - \frac{4c_0 \ell^{(1)}(\beta)}{(\alpha+\beta)^2} + O\left(\frac{1}{\alpha+\beta}\right)^3 \right] + O(\varepsilon^2), \qquad (2.111)$$

$$x - c_0 t = H^{(0)}(\beta) + \varepsilon H^{(1)}(\beta) + \varepsilon^2 H^{(2)}(\beta) + \left[\varepsilon \ell^{(1)}(\beta) + \varepsilon^2 \ell^{(2)}(\beta) + \ldots \right] \cdot \left[\frac{2}{\alpha+\beta} - \frac{\varepsilon 4x^{(1)}}{(\alpha+\beta)^2} + \ldots \right] +$$

$$+ \varepsilon Z^{(1)}(\beta) \log \frac{\alpha+\beta}{2} + \varepsilon Z^{(1)}(\beta) \left[\frac{2\varepsilon x^{(1)}}{\alpha+\beta} + \ldots \right] + \varepsilon^2 Z^{(1)2} \left[\frac{1}{2} K \log \frac{\alpha+\beta}{2} - K_2 \right] \left[\frac{2}{\alpha+\beta} + O(\varepsilon) \right]$$

$$(2.112)$$

Then upon comparing these results with ours, it is found that (2.111) agrees with (2.105) and (2.112) agrees with (2.106) provided

$$H^0(z) = \beta$$

$$H^{(1)}(\beta) = \tilde{x}_1(\beta) + \log 2$$

$$2c_0 \ell^{(1)}(\beta) = g(\beta) \qquad (2.113)$$

$$Z^{(1)}(\beta) = \frac{\gamma+1}{4c_0} \frac{dg(\beta)}{d\beta} .$$

This is a consistent set of relations since by differentiating (2.109) one finds

$$kzh'(z) = \varepsilon \frac{d\ell^{(1)}}{dz} + \ldots, \qquad (2.114)$$

and then using (2.110) and (2.113)

$$\frac{d\mathcal{J}^{(1)}}{d\beta} = \frac{d\mathcal{J}^{(1)}}{dz}\frac{dz}{d\beta} = \frac{d\mathcal{J}^{(1)}}{dz}\frac{1}{h'(z)},$$

so that

$$kzh'(z) = k\varepsilon Z^{(1)}(\beta)h'(z) = k\varepsilon\left[\frac{\gamma+1}{4c_o}\frac{dg(\beta)}{d\beta}\right]h'(z) = \frac{\varepsilon}{2c_o}\frac{dg(\beta)}{d\beta}h'(z)$$

and

$$\varepsilon\frac{d\mathcal{J}^{(1)}(\beta)}{dz} = \varepsilon h'(z)\frac{d\mathcal{J}^{(1)}}{d\beta} = \frac{\varepsilon h'(z)}{2c_o}\frac{dg(\beta)}{d\beta} = \frac{\varepsilon}{2c_o}\frac{dg(\beta)}{d\beta}h'(z),$$

and (2.114) holds under the conditions of (2.113).

Thus the first order solutions are in agreement, and although we do not have higher order solutions available for comparison, it is possible to show that Whitham's equations asymptotically take on the form (2.101) estimated in section 1. Asymptotically his equations (12) and (13) become

$$u \sim zO(\tfrac{1}{x}) + z^2O(\tfrac{\log x}{x^2}) + \ldots$$
$$c_ot - x \sim zO(\log x) + z^2O(\tfrac{\log x}{x}) + \ldots$$

in good agreement with our estimates

$$u \sim \varepsilon O(\tfrac{1}{\beta}) + \varepsilon^2O(\tfrac{\log\beta}{\beta^2}) + \ldots$$

$$x \sim O(\beta) + \varepsilon O(\log\beta) + \varepsilon^2O(\tfrac{\log\beta}{\beta}) + \ldots .$$

Evidently a good deal of work remains to be done in the application of coordinate perturbation to spherical (and cylindrical) wave propagation problems. Not only would general convergence

proofs be desirable for a process involving both sets of characteristics, but even an investigation of the form of the next order perturbation solutions would be useful. Particular problems could be studied with special reference to those involving the development of shock waves so that the perturbation solution could be compared with the solution obtained by other means (e.g., numerical) both in the ordinary regions of the flow and in the neighborhood of the shock.

THE ELLIPTIC CASE

Introduction

The applicability of the coordinate perturbation type of solution to problems of an elliptic nature has been studied by investigating the problem of the flow past a thin airfoil. In the usual thin airfoil theory the solution obtained tends to break down near the leading edge of the airfoil and so in particular one would wish to correct the solution in this neighborhood. Lighthill considered the problem for the incompressible case [11] and found that the first order solution could be rendered uniformly valid by imparting a constant shift to the coordinate system. Furthermore he implied that for the higher order solutions any tendency for singularities to arise could be suppressed by finding suitable higher order perturbation functions for the coordinate system.

In this chapter, however, it is shown that the perturbation functions required to render the solution valid at each stage have increasing orders of singularity near the leading edge of the airfoil so that they do not represent a satisfactory coordinate transformation. Thus although a one-stage correction does effect an improvement in the uniform validity of the velocity fields for the incompressible case, the general applicability of the method at the next stages cannot be affirmed.

As far as the compressible case is concerned, even the one-stage correction proves unsatisfactory, and so in general it does not seem possible at this point to recommend the method of coordinate perturbation as a panacea for problems of an elliptic character.

Section 1: Incompressibe case: Difficulties with Lighthill's solution.

In the following it will be shown that if a coordinate perturbation of the type

$$x = X + \varepsilon x^{(1)}(X,Y) + \varepsilon^2 x^{(2)}(X,Y) + \varepsilon^3 x^{(3)}(X,Y) + \ldots$$
$$y = Y \tag{3.1}$$

is to be used to correct the perturbation solution to the problem of incompressible flow past a body whose leading edge at $x = 0$ behaves like

$$y = \varepsilon x^s \tag{3.2}$$

Then the process will be unsuccessful for $s < 1$. It will happen that near $x = 0$ the perturbation functions, $x^{(1)}$, have the behavior:

$$x^{(2)} \sim x^{2s-1}, \quad x^{(3)} \sim x^{(2)}, \quad x^{(4)} \sim x^{4s-3}, \quad x^{(5)} \sim x^{(4)}, \text{ etc.} \tag{3.3}$$

so that all the functions will be positive powers of x as $x \to 0$, only for

$$\lim_{n\to\infty} \quad s \geqslant \frac{n-1}{n} \text{ or } s \geqslant 1.$$

Lighthill [11] took the case $s = 1/2$ so that his $x^{(2)}$ is a constant, but his higher order perturbation functions become singular and the process breaks down.

In a general application of a coordinate transformation of the type (3.1), the perturbation functions for the y-coordinate would have to be restored, and the differential equations and boundary conditions on both sets of functions $x^{(1)}$ and $y^{(1)}$ determined. In this particular problem it may be shown that the equations can be satisfied by setting the pair $x^{(1)}$ and $y^{(1)}$ to be harmonic conjugate functions, but that it is also possible to set $y = Y$ as Lighthill does, provided that higher order perturbations in the velocities (potential) are retained. The matter of greater interest is the boundary condition, and in fact Lighthill determines the function $x^{(2)}$ solely on this basis.

In his equation (28), [11] he sets $u(x,y) \rightarrow u(X,y)$ and $v(x,y) \rightarrow v(X,y)$, where u and v are the velocity components in the x and y directions respectively, and then takes

$$\frac{v(X,y)}{u(X,y)} = F'(X + \varepsilon^2 x^{(2)}) \text{ on the airfoil defined}$$

by
$$y = \varepsilon F(X + \varepsilon^2 x^{(2)}),$$

where for simplicity we have taken the camber equal to zero, and have further neglected the function $x^{(1)}$ agreeing with Lighthill that it is not necessary. The point of the step is to improve the usual perturbation solution $(u(x,y),\ v(x,y))$, which is not valid uniformly, by expressing the solution in terms of new coordinates in an X-y plane and then by determining the appropriate coordinate transformation between this plane and the physical plane which will send the approximate

solution into a uniformly valid one. Now in view of this mapping idea it seems simpler to express the problem in terms of a stream function since in the first place the image solution in the (x,y)-plane can be clearly visualized, and secondly, only one dependent perturbation solution is required. Using the stream function it is possible to duplicate the steps and motivation of Lighthill's paper and to show how difficulties occur at the next order perturbation which he did not consider.

For the stream function, Ψ, where $u = \Psi_y$, $v = -\Psi_x$, the boundary conditions become

$$\frac{v}{u} = \frac{-\Psi_x}{\Psi_y} = \frac{dy}{dx} = \varepsilon F'(x) \text{ on } \begin{cases} y = \varepsilon F(x) \\ 0 \leq x \leq c \end{cases}$$

$(x=0)$ = leading edge

$(x=c)$ = trailing edge

$$\text{or } \Psi(x,y) = \text{constant} = 0. \tag{3.4}$$

The analogue of Lighthill's equation (28) is then

$$\Psi(X,y) = \text{constant} = 0 \quad \text{on} \quad y = \varepsilon F(x) = \varepsilon F(X + \varepsilon^2 x^{(2)} + \ \ldots). \tag{3.5}$$

Now for

$$\Psi = y + \varepsilon\, \psi^{(1)}(X,y) + \varepsilon^2\, \psi^{(2)}(X,y) + \ldots$$

and for

$$\psi^{(1)}(X,y) = \psi^{(1)}(X,0) + y\frac{\partial \psi^{(1)}(X,0)}{\partial y} + \frac{y^2}{2}\frac{\partial^2 \psi^{(1)}}{\partial y^2} + \ldots, \quad y \text{ small},$$

and for

$$y = \varepsilon F(X + \varepsilon^2 x^{(2)} + \ldots) = \varepsilon F(X) + \varepsilon^3 x^{(2)} F'(X) + \ldots,$$

one finds, as he did, a set of boundary equations of the

following types, derived from setting the coefficients of powers of ε equal to zero.

Boundary conditions

Terms order ε $\quad \psi^{(1)}(x,0) = -F(x) \qquad\qquad 0 \leq x \leq c$

$$\text{Terms order } \varepsilon^2 \quad \psi^{(2)}(x,0) = -F(x)\frac{\partial \psi^{(1)}}{\partial y}\Big]_{x,0}$$

$$\text{Terms order } \varepsilon^3 \quad \psi^{(3)}(x,0) = -F\frac{\partial \psi^{(2)}}{\partial y} - x^{(2)}F' - \frac{F^2}{2}\frac{\partial^2 \psi^{(1)}}{\partial y^2} \tag{3.6}$$

$$\text{Terms order } \varepsilon^4 \quad \psi^{(4)}(x,0) = -F\frac{\partial \psi^{(3)}}{\partial y} - x^{(3)}F' - x^{(2)}F'\frac{\partial \psi^{(1)}}{\partial y} - \frac{F^2}{2}\frac{\partial^2 \psi^{(2)}}{\partial y^2} - \frac{F^3}{3!}\frac{\partial^3 \psi^{(1)}}{\partial y^3}$$

$\vdots$

In these equations the arbitrary functions, $x^{(2)}$, $x^{(3)}$, etc. are to be chosen to suppress increasing orders of singularity (near $x = 0$) appearing in the successive $\psi^{(i)}$'s. It is necessary to estimate the order of the terms involved and we do this for a general case

$$F(x) = x^s(F_0 + F_1x + \ldots), \qquad s > 0.$$

From the terms of order

$$\psi^{(1)}(x,0) = -F(x) = -x^s(F_0 + F_1x + \ldots),$$

so that the completed complex potential function,

$$G^{(1)} = \varphi^{(1)} + i\,\psi^{(1)},$$

which is to be used for finding $\frac{\partial \psi^{(1)}}{\partial y}$ in this neighborhood behaves like

$$G^{(1)} = -iF_0z^s + \ldots \qquad , \; z = x + iy.$$

Then from

$$\frac{dG^{(1)}}{dZ} = \frac{\partial\psi^{(1)}}{\partial y} + i\frac{\partial\psi^{(1)}}{\partial X} = -iF_0 sZ^{s-1} + \ldots$$

one finds $\frac{\partial\psi^{(1)}}{\partial y}$ is regular in the neighborhood (as $y \to 0$, $X \to 0$). Similarly

$$\frac{d^2G^{(1)}}{dZ^2} = \frac{\partial^2\psi^{(1)}}{\partial X\partial y} + i\frac{\partial^2\psi^{(1)}}{\partial X^2} = -iF_0 s(s-1)Z^{s-2} + O(Z^{s-1})$$

or

$$\lim_{y\to 0} \frac{\partial^2\psi^{(1)}}{\partial y^2} = -\frac{\partial^2\psi^{(1)}}{\partial X^2} \longrightarrow F_0 s(s-1)X^{s-2} + \ldots$$

and in turn

$$\lim_{y\to 0} \frac{\partial^3\psi^{(1)}}{\partial y^3} = O(X^{s-2}).$$

Using these evaluations in equations (3.6) shows that whereas $\psi^{(2)}$ is finite as well as $\psi^{(1)}$ on the boundary (both $\sim X^s$), an increase in order of singularity appears in the ε^3 equation due to the term:

$$-\frac{F^2}{2}\frac{\partial^2\psi^{(1)}}{\partial y^2} \sim -\frac{F_0{}^2X^{2s}}{2} \cdot F_0 s(s-1)X^{s-2} = \frac{F_0{}^3 s(s-1)X^{3s-2}}{2}.$$

$x^{(2)}$ is to be chosen to cancel out this higher order singularity, or

$$-x^{(2)}F' - \frac{F^2}{2}\frac{\partial^2\psi^{(1)}}{\partial y^2} = 0$$

$$x^{(2)} = -\frac{F_0{}^2(s-1)}{2}X^{2s-1}.$$

Then from the terms of the ε^4 equation (3.6), since $\frac{\partial^3\psi^{(1)}}{\partial y^3}$ is not more singular than $\frac{\partial^2\psi^{(2)}}{\partial y^2} \sim \frac{\partial^2\psi^{(1)}}{\partial y^2}$, the most singular terms involve

$$-x^{(3)}F' - \frac{F^2}{2}\frac{\partial^2\psi^{(2)}}{\partial y^2} = 0$$

giving $\quad x^{(3)} = 0(X^{2s-1})$, which is not more singular than $x^{(2)}$.

However, the equation of terms of order ε^5 will contain a term

$$\frac{F^4}{4!}\frac{\partial^4\psi^{(1)}}{\partial y^4} = 0(X^{5s-4})$$

to be balanced by a term, $-x^{(4)}F'$ so that

$$x^{(4)} = 0(X^{4s-3}).$$

In summary then for

$$x = X + \varepsilon^2 x^{(2)}(X,y) + \varepsilon^3 x^{(3)}(X,y) + \ldots$$

as $y \to 0$

$X = 0^+$ $\qquad\qquad\qquad\qquad\qquad\qquad$ for $s = 1/2$

$$\begin{aligned}
x^{(2)} &= -\frac{F_0{}^2(s-1)}{2}X^{2s-1} + \ldots &\qquad x^{(2)} &= \frac{F_0^2}{4}\\
x^{(3)} &= 0(X^{2s-1}) & x^{(3)} &= 0(1)\\
x^{(4)} &= 0(X^{4s-3}) & x^{(4)} &= 0(\tfrac{1}{X}) \qquad (3.7)\\
x^{(5)} &= 0(X^{4s-3}) & x^{(5)} &= 0(\tfrac{1}{X})\\
x^{(6)} &= 0(X^{6s-5}) & x^{(6)} &= 0(\tfrac{1}{X^2})
\end{aligned}$$

Thus Lighthill's determination that for $s = 1/2$, $x^{(2)} = \frac{F_0^2}{4}$, is quite correct and the correction serves to reduce the error in velocity at the leading edge from O(1) to $O(\varepsilon)$.* However unfortunately the usual perturbation scheme is such that each higher order perturbation function, $\psi^{(i)}$, introduces a higher order singularity in this region (c.f. equations (3.6)), and the coordinate perturbation functions, $x^{(i)}$, cannot suppress the singularity without carrying it themselves. The result is that near the leading edge the perturbation series,

$$x = X + \varepsilon^2 x^{(2)}(X,0) + \varepsilon^3 x^{(3)}(X,0) + \varepsilon^4 x^{(4)} + \varepsilon^5 x^{(5)} + \varepsilon^6 x^{(6)} + \ldots$$

evaluated say for $X = O(\varepsilon^2)$, (which holds at the image in the (X,y)-plane of the leading edge), becomes (for $s = 1/2$),

$$x = O(\varepsilon^2) + O(\varepsilon^2) + O(\varepsilon^3) + O(\varepsilon^2) + O(\varepsilon^3) + O(\varepsilon^2) + \ldots$$

and the perturbation is no longer valid.

Also for $s \neq 1/2$ from (3.7), the perturbation functions will be positive powers of X only for $s \geqslant 1$, that is for profiles with a finite (or zero) slope at the origin, and so a convergent scheme can be evolved only for special simple cases where the leading edge is not blunt. Thus at this stage of the investigation no way seems available for extending the method to a generally useful scheme.

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* The improvement in the solution for velocity is demonstrated in the supplement to this section.

A few further points involved in the failure of the method are discussed in the next section with particular reference to the problem of the incompressible flow past an elliptic cylinder.

Supplement to section 1: Improvement of the solution due to coordinate shift

In the following is shown the improvement in the solution for the velocity near the leading edge (at $x = 0$) effected by a constant shift in the coordinate system. The airfoil is assumed to have the profile

$$y = \varepsilon F(x) = \varepsilon \left[F_0 \sqrt{x} + F_1 x^{3/2} + \ldots\right], \quad 0 \leq x \leq c. \tag{3.8}$$

If a usual perturbation solution is found for the problem and is expressed in terms of new variables X and Y, the streamline $\Psi = 0$ becomes,

$$\Psi = Y + \frac{\varepsilon}{\pi} \mathcal{I}m \int_0^c \frac{F(w)\,dw}{Z-w} = 0, \tag{3.9}$$

or, for the profile itself

$$1 = \frac{\varepsilon}{\pi} \int_0^c \frac{F(w)\,dw}{(X-w)^2 + Y^2}. \tag{3.10}$$

Now since the profile given by (3.10) differs from that given by (3.8), an error in velocity will appear so that at corresponding points, z_A on the actual profile (3.8) and Z_s on the solution (3.10) there will be velocity discrepancy of size

$$(z_A - Z_s)\frac{\partial}{\partial Z}(\text{velocity}) \tag{3.11}$$

In (3.11) so long as the rate of change of velocity

is of order one, a difference in profiles of order ε^2 will introduce a velocity error of order ε^2 only. However near the leading edge as was shown earlier in the section,

$$O\left(\frac{\partial}{\partial z}(\text{velocity})\right) = O\left(\frac{\partial^2 \psi}{\partial z^2}\right) = O\left(\varepsilon \frac{\partial^2 \psi^{(1)}}{\partial Z^2}\right) = O\left(\varepsilon \frac{1}{Z^{s-2}}\right),$$

so that for a boundary difference of order ε^2 (with $s = 1/2$),

$$O\left(\frac{\partial}{\partial z}(\text{velocity})\right) = O\left(\varepsilon \frac{1}{(\varepsilon^2)^{3/2}}\right) = O\left(\frac{1}{\varepsilon^2}\right) ,$$

and from (3.11) the boundaries must agree through order ε^2 to give accuracy even up to order ε in velocity.

Thus it is to be shown that a coordinate correction of the type,

$$\begin{aligned} x &= X + \varepsilon^2 x^{(2)} + \ldots &&= X + \varepsilon^2 \frac{F_0^2}{4} + \cdots \\ y &= Y &&= Y \end{aligned} \tag{3.12}$$

will bring the actual profile and the solution profile into agreement at least through order ε^2 in the neighborhood of the leading edge. Away from this region, since the velocity derivative becomes of order one, a constant shift of order ε^2 will give only an ε^2-order error in velocity.

Near the leading edge on the actual airfoil let there be a parameter k such that

$$x = k^2 \varepsilon^2$$

$$\text{and then} \qquad y = \varepsilon F_0 \sqrt{x} = \varepsilon^2 F_0 k. \tag{3.13}$$

Then the same coordinates are to hold through order ε^2 for corresponding points on the streamline of (3.10). In the integral of this equation, letting $Y = y = F_0 k \varepsilon^2$ and $w = t^2$, one finds

$$\int_0^{\sqrt{c}} \frac{t^2 dt}{t^4 - 2t^2 X + X^2 + F_0^2 k^2 \varepsilon^2} = \int_0^{\sqrt{c}} \frac{t^2 dt}{t^4 + bt^2 + c^2} = \int_0^{\sqrt{c}} \frac{B dt}{t^2 + t_1^2} + \int_0^{\sqrt{c}} \frac{D dt}{t^2 + t_1^2} \qquad (3.14)$$

where

$$b = -2X \qquad\qquad B = \frac{(b + \sqrt{b^2 - 4c})^2}{(b + \sqrt{b^2 - 4c})^2 - 4c}$$

$$c = X^2 + F_0^2 k^2 \varepsilon^2 \qquad\qquad D = \frac{-4c}{(b + \sqrt{b^2 - 4c})^2 - 4c}$$

$$t_1^2 = \frac{b + \sqrt{b^2 - 4c}}{2}$$

$$t_2^2 = \frac{b - \sqrt{b^2 - 4c}}{2}$$

Then since

$$\int_0^{\sqrt{c}} \frac{B dt}{t^2 + t_1^2} = \frac{B}{t_1} \tan^{-1} \frac{\sqrt{c}}{t_1} \quad ,$$

and since t_1, t_2 are order $\sqrt{X}$ which is order $\sqrt{\varepsilon^2}$ in the region under consideration,

$$\int_0^{\sqrt{c}} \frac{B dt}{t^2 + t_1^2} = \frac{B}{t_1} \left[\frac{\pi}{2} - \frac{t_1}{\sqrt{c}} - \cdots \right] = \frac{B}{t_1} \left[\frac{\pi}{2} + O(\varepsilon) \right] ,$$

and thus

$$\frac{2\varepsilon F_0}{\pi}\int_0^{\sqrt{c}} \frac{t^2 dt}{t^4-2k^2X+X^2+F^2k^2\varepsilon^2} = \varepsilon F_0\left[\frac{B}{t_1}+\frac{D}{t_2}+O(1)\right] . \quad (3.15)$$

From (3.12), in the region defined by (3.13)

$$X = x - \frac{\varepsilon^2 F_0^2}{4} = \varepsilon^2(k^2 - \frac{F_0^2}{4}), \quad (3.16)$$

and this value for X together with $y = Y = \varepsilon^2 F_0 k$ must yield a point on the streamline (3.10) at least through order ε^2 accuracy. From (3.15) since $t_1, t_2 = O\sqrt{x}$, an error of ε^2 in (3.16) would cause an error of order one in (3.10), and so it need only be shown that (3.16) as given allows (3.10) to hold through order one. By straight substitution one obtains

$$\varepsilon F_0\left[\frac{B}{t_1}+\frac{D}{t_2}+\ldots\right] = \frac{1}{8ik}\left[4(\frac{F_0}{2}+ik) - 4(\frac{F_0}{2}-ik)\right] = \frac{8ik}{8ik} = 1$$

as required.

In other words near the leading edge the boundaries differ by $O(\varepsilon^3)$ only; away from this region this discrepancy is at most $O(\varepsilon^2)$, and the velocity error has been smoothed to be at most $O(\varepsilon)$ in the entire vicinity.

Section 2: Incompressible flow past an elliptic cylinder

In the previous section we saw that each new perturbation function $\psi^{(i)}$, introduced an increase in order of singularity near the leading edge. One is led to inquire then if it would not be possible to use only the terms

$$\psi = \psi^{(0)} + \varepsilon \psi^{(1)},$$

and to throw the solution correction entirely into a perturbation in both x and y coordinates. In fact it may be shown that the perturbation functions $y^{(1)}(X,Y)$ always appear in the equations associated with the respective $\psi^{(1)}$ functions, so that either set of functions may be used alone. Unfortunately this reciprocity also implies that the process will fail in these new terms, as indeed it does, but some facts of interest arise from the investigation.

The elliptic cylinder has been studied in order that intermediate approximate solutions could be compared with the exact solution. For the ellipse,

$$x^2 + \frac{y^2}{\varepsilon^2} = 1, \qquad \varepsilon \ll 1, \tag{3.17}$$

the exact solution for the complex potential, F, is

$$F = \frac{U}{1-\varepsilon}\left[z - \varepsilon\sqrt{z^2 - 1 + \varepsilon^2}\right], \tag{3.18}$$

where $z = x + iy$

$$F = \varphi + i\psi \begin{cases} \varphi = \text{potential function} \\ \psi = \text{stream function} \end{cases}$$

U = uniform flow at infinity.

Then, attempting a perturbation solution,

$$\tilde{F} = \tilde{F}^{(0)} + \varepsilon \tilde{F}^{(1)}(X,Y),$$

an approximate solution may be found for the complex potential

$$\tilde{F} = AU\left[z + \varepsilon(z - \sqrt{z^2 - 1})\right] \tag{3.19}$$

where A is a constant depending on any scale changes entering, and where Z has been used in the sense that the capital letter coordinates were used in section 1.

The streamline arising from (3.19) as representing the airfoil is

$$X^2 + \frac{Y^2(1+\varepsilon)^2}{2} = \frac{(1+\varepsilon)^2}{1+2\varepsilon}\,. \tag{3.20}$$

Thus when the (x,y)-plane and the (X,Y)-plane are superimposed, a discrepancy shows up between the airfoil streamline in the two planes.

The question to be investigated is whether a coordinate perturbation of the form

$$z = Z + \varepsilon z^{(1)}(Z) + \varepsilon^2 z^{(2)}(Z) + \ldots \tag{3.21}$$

can be used to resolve the difference between the two streamlines and thus make (3.19) a solution of the problem. For the ellipses of (3.17) and (3.20) the mapping is given by

$$z = \frac{1}{\sqrt{1+2\varepsilon}}\left[Z(1+\varepsilon-\varepsilon^2) + \varepsilon^2\sqrt{Z^2-1}\right], \tag{3.22}$$

and since this can be expressed in the form of equation (3.21)

$$z = Z + \varepsilon^2\left(-\frac{Z}{2} + \sqrt{(Z^2-1)}\right) - \varepsilon^3\sqrt{Z^2-1} + \varepsilon^4\left(\frac{3}{8}Z + \frac{3}{2}\sqrt{Z^2-1}\right) + \ldots \tag{3.23}$$

one is led to hope that the perturbation idea will work for the problem.

In order to find the solution (3.23) in the general case one would take the perturbations,

$$x = X + \varepsilon x^{(1)}(X,Y) + \varepsilon^2 x^{(2)}(X,Y) + \ldots$$
$$y = Y + \varepsilon y^{(1)}(X,Y) + \varepsilon^2 y^{(2)}(X,Y) + \ldots \qquad (3.24)$$

and find from the equations that $x^{(1)}$ and $y^{(1)}$ are harmonic conjugate functions whose boundary conditions are to be set by a perturbation mapping of (3.17) into (3.20). As far as the first order perturbation functions, $x^{(1)}$ and $y^{(1)}$, are concerned, it may easily be shown correct to take

$$x^{(1)} = y^{(1)} \equiv 0.$$

The equation for the agreement of boundaries then becomes,

$$4\left[\varepsilon^2 y^{(2)2} + 2\varepsilon^3 y^{(2)} y^{(3)} + \ldots\right]\left[(1-X^2) - 2\varepsilon(1-X^2) + \varepsilon^2(1+3(1-X^2)) + \ldots\right] =$$
$$\left\{2\varepsilon(1-X^2) - \varepsilon^2\left[1+3(1-X^2+2Xx^{(2)}+y^{(2)2}\right] + \ldots\right\}^2, \qquad (3.25)$$

and of course another condition must be specified to determine both $y^{(2)}$ and $x^{(2)}$. For the case $(1-X^2) \not< O(\varepsilon^2)$ one obtains

$$y^{(2)} = \sqrt{1-X^2}, \qquad (3.26)$$

but for the case $(1-X^2) = O(\varepsilon^2)$ by expanding both $x^{(2)}$ and $y^{(2)}$ as power series around an extremum of the streamline ellipse (3.20) one obtains, in terms of the local variables

$$\xi = \frac{X - \left(\frac{1+\varepsilon}{\sqrt{1+2\varepsilon}}\right)}{\varepsilon^2}$$

$$\eta = \frac{Y}{\varepsilon^2}$$

the series for harmonic conjugate functions in the form

$$x^{(2)} = a_{00} + a_{10}\xi + a_{20}\xi^2 - a_{20}\eta^2 + a_{30}\xi^3 + \ldots$$

$$y^{(2)} = \qquad a_{10}\eta + 2a_{20}\xi\eta + a_{20}3\eta\xi^2 - a_{30}\eta^3 + \ldots \quad . \tag{3.27}$$

Then by imposing regularity on $x^{(2)}$ and $y^{(2)}$ at infinity (in the $\frac{Y}{\varepsilon^2}$ scale) one finds of course that only a constant fulfills the requirements, so that exactly as in section 1, the required coordinate change is

$$x^{(2)} = -\frac{1}{2}$$

$$y^{(2)} = 0. \tag{3.28}$$

However, the disagreement of this local condition (3.28) with the condition (3.26) shows that a general boundary condition has not been found, and of course the functions $x^{(2)}$ and $y^{(2)}$ are not determined until their values along the entire boundary are given. That such a general boundary condition cannot be found by the perturbation method we are using is unfortunately true as can be shown in the following way.

The correct perturbation function of second order is given by the known solution of (3.23) as

$$x^{(2)} + iy^{(2)} = z^{(2)} = -\frac{z}{2} + \sqrt{z^2 - 1} \; . \tag{3.29}$$

When this is evaluated on the boundary given by (3.20) one finds for example

$$\left(y^{(2)} \right)^2 = \left(\frac{1+\frac{\varepsilon}{2}}{1+\varepsilon} \right)^2 \left(1 - x^2 + \frac{\varepsilon^2}{1+2\varepsilon} \right), \tag{3.30}$$

and so it is shown that the unique correct function $y^{(2)}$ which is specified uniquely by the boundary condition (3.30) must have that boundary condition a function of ε. Any perturbation method which requires the conditions to be free from ε cannot give a correct result.

In summary, then, the attempt to improve an approximate solution near the leading edge of the airfoil fails beyond the first step since the boundary conditions on the perturbation functions must involve ε which contradicts the perturbation method itself. Although it is true that the one-step correction involving a constant shift can reduce the large error in the velocity at the leading edge, no clear cut method for extending the process seems to present itself.

One further attempt at coordinate perturbation was attempted and will be described briefly. It seemed possible that the difficulty with the boundary conditions near the leading edge was due to the "sharpness" or rapid change of slope in this region, and could be avoided by mapping the streamline of the approximate solution into a more nearly circular form. The process was carried out, but proved unsuccessful from the perturbation point of view.

The steps followed were first the determination of the mapping of the approximate streamline (3.20) into a circle in the w plane. In this step the mapping

$$Z = \frac{1}{2}\left(w + \frac{1}{w}\right) \tag{3.31}$$

takes (3.20) into a circle

$$|w| = \sqrt{1+2\varepsilon} = 1+\varepsilon - \frac{\varepsilon^2}{2} + \dots \quad .$$

Then the image of the actual airfoil (3.17) under this mapping (3.31) was found. For $w = \rho e^{i\theta}$ (3.17) becomes

$$\rho^2 = \frac{\sqrt{\varepsilon^2 + \sin^2\theta} + \varepsilon\sin\theta}{\sqrt{\varepsilon^2 + \sin^2\theta} - \varepsilon\sin\theta} . \tag{3.32}$$

The question then is raised as to whether ρ may be expressed as a perturbation series

$$\rho = 1 + \varepsilon g(\theta) + \dots \tag{3.33}$$

where $g(\theta)$ and its derivatives are of order one near the region in question ($\theta \sim 0$).

However

$$2\rho\rho_\theta = \frac{2\varepsilon^3\cos\theta}{\sqrt{\varepsilon^2 + \sin^2\theta}\;\left(\sqrt{\varepsilon^2 + \sin^2\theta} - \varepsilon\sin\theta\right)^2} \tag{3.34}$$

and near $\theta = 0$, $\rho = 0(1)$ so

$$2\rho\rho_\theta = 0(1)$$
$$\rho_\theta = 0(1),$$

and

$$\frac{dg}{d\theta} = \frac{1}{\varepsilon}\frac{d}{d\theta}(\rho - 1) = \frac{1}{\varepsilon}\,0(1) = 0\left(\frac{1}{\varepsilon}\right). \tag{3.35}$$

Thus the derivatives of g change order and an expansion of the type (3.33) is not possible even in these terms. Actually g is a function of ε in this region, and in fact if one sets

$$\frac{\sin\theta}{\varepsilon} = c \tag{3.36}$$

then (3.32) may be expressed as

$$\rho^2 = 1 + 2\frac{c\varepsilon}{\sqrt{1+c^2}} + 2\left(\frac{c\varepsilon}{\sqrt{1+c^2}}\right)^2 + 2\left(\frac{c\varepsilon}{\sqrt{1+c^2}}\right)^3 + \ldots \tag{3.37}$$

where the function $\frac{c\varepsilon}{\sqrt{1+c^2}}$ remains $O(\varepsilon)$ for the full range $0 \leq c \leq \frac{1}{\varepsilon}$.

It seems clear that the method of coordinate perturbation as it has been used in this study is not satisfactory as it stands for improving solutions to elliptic problems. Clearly if one takes a coordinate perturbation of the form

$$z = Z + \varepsilon g^{(1)}(Z) + \varepsilon^2 g^{(2)}(Z) + \ldots \tag{3.38}$$

under the requirements that

(i) $\frac{dz}{dZ} = 1$ or constant as $Z \to \infty$

(ii) $g^{(1)}(Z)$ be regular outside the body

(iii) $g^{(1)}(Z)$ be of order 1 in some small region (say of order ε^2) near the body ($Z = O(\varepsilon^2)$),

then

$$g^{(1)}(Z) = Z\left(c_0 + \frac{c_1}{Z} + \frac{c_2}{Z^2} + \ldots\right) \to Z\left(c_0 + \frac{b_1\varepsilon^2}{Z} + \frac{c_2\varepsilon^4}{Z^2} + \ldots\right)$$

or $\frac{g^{(1)}(Z)}{Z}$ = function of $\frac{Z}{\varepsilon^2}$ which contradicts an expansion of the perturbation type (3.38).

Unfortunately the study of the incompressible case, which was undertaken in the hope that some method of solution improvement might be found which could carry over to the compressible case, has not yielded any such method. The incompressible case retains its former status as a straight mapping problem, and it will be found that even the constant coordinate shift which served as a one-stage velocity correction in this work cannot be applied satisfactorily to the compressible case.

Section 3: The compressible case

In the hope of finding at least a one-stage correction to the velocity given by the usual perturbation solution to the compressible flow past a thin airfoil, the problem was investigated again for the case of an elliptic cylinder. For compressible flow however the equations for the different order coordinate perturbation functions are no longer simple Laplace equations, but involve nonhomogeneous terms based on lower order functions. Further, these terms contribute increasing orders of singularity near the leading edge of the airfoil. Thus where in the incompressible case the coordinate perturbation functions were chosen to suppress singularities arising from the boundary conditions, in the compressible case the choice is governed by suppression of singularities in the equations. In this case it will be found just as before that the increase in order of singularity

of the perturbation functions themselves prevents the perturbation series from converging near the leading edge, but even worse in this case not even a first stage constant shift is permissible since it does not satisfy the equations involved.

That this is so may be seen by a study of the equations of the problem. The equation for the streamfunction, Ψ, for compressible flow is

$$\Psi_{xx}\left(\frac{c^2}{c_0^{\ 2}} - M^2u^2\right) - 2M^2uv\Psi_{xy} + \Psi_{yy}\left(\frac{c^2}{c_0^{\ 2}} - M^2v^2\right) = 0 , \qquad (3.39)$$

where $\rho u = \Psi_y$

$-\rho v = \Psi_x$

M = Mach number

c_0 = speed of sound at infinity

ρ = density.

The problem studied here has been simplified to the extent of using the special case

$$\text{pressure, } P = \alpha - \frac{\beta^2 c_0^{\ 2}}{\rho} , \quad \alpha, \beta^2 = \text{constants}, \qquad (3.40)$$

so that (3.39) becomes

$$\Psi_{xx}\left(1 - \frac{M^2}{\beta^2}\Psi_y^2\right) + \frac{2M^2}{\beta^2}\Psi_x\Psi_y\Psi_{xy} + \Psi_{yy}\left(1 - \frac{M^2}{\beta^2}\Psi_x^{\ 2}\right) = 0. \qquad (3.41)$$

Then solving the problem on the ellipse (3.17)

$$x^2 + \frac{y^2}{\varepsilon^2} = 1$$

by the usual perturbation method, and then expressing the solution in terms of new variables, X and Y, exactly as was done for the incompressible case, one finds that a solution

$$\Psi = Y + \varepsilon \Psi^{(1)}(X,Y) \tag{3.42}$$

gives the following streamline as an approximation to the ellipse,

$$X^2 + \frac{Y^2}{\varepsilon^2}(1 + \varepsilon k)^2 = 1 + \frac{\varepsilon^2 k^2}{1 + 2\varepsilon k} \tag{3.43}$$

where $\quad k^2 = 1 - \frac{M^2}{\beta^2}$.

To correct the solution a coordinate perturbation of the type

$$\begin{aligned} x &= X + \varepsilon x^{(1)} + \varepsilon^2 x^{(2)} + \ldots \\ y &= Y \qquad\quad + \varepsilon^2 y^{(2)} + \ldots \end{aligned} \tag{3.44}$$

is attempted. (Only one of the functions, $\Psi^{(i)}$ or $y^{(i)}$ need be used because they are redundant. We have taken $\Psi^{(i)} \equiv 0$, $i > 1$.)

Substitution of (3.44) into (3.41) yields a set of equations based on equating to zero the coefficients of each power of ε . The equations are

<u>Terms of order ε</u>

$$(1 - \frac{M^2}{\beta^2})\Psi^{(1)}_{XX} + \Psi^{(1)}_{YY} = 0 \tag{3.45}$$

which is satisfied by the usual first order perturbation function

<u>Terms of order ε^2</u>

$$\begin{aligned} (1 - \frac{M^2}{\beta^2}) y^{(2)}_{XX} + y^{(2)}_{YY} = & -\Psi^{(1)}_X \left[(1 - \frac{M^2}{\beta^2}) x^{(1)}_{XX} + x^{(1)}_{YY} \right] - \\ & 2\left[(1 - \frac{M^2}{\beta^2}) x^{(1)}_X \Psi^{(1)}_{XX} + x^{(1)}_Y \Psi^{(1)}_{XY} \right] + \\ & \frac{2M^2}{\beta^2}\left[\Psi^{(1)}_X \Psi^{(1)}_{XY} - \Psi^{(1)}_Y \Psi^{(1)}_{XX} \right] . \end{aligned} \tag{3.46}$$

Terms of order ε^3

$$
\begin{aligned}
(1-\tfrac{M^2}{\beta^2})y^{(3)}_{XX} + y^{(3)}_{YY} = &-\psi^{(1)}_{X}\left[(1-\tfrac{M^2}{\beta^2})x^{(2)}_{XX} + x^{(2)}_{YY}\right] - 2x^{(2)}_{Y}\psi^{(1)}_{XY} + \\
&+ x^{(2)}_{X}\left[3(1-\tfrac{M^2}{\beta^2})\psi^{(1)}_{XX} + 5\psi^{(1)}_{YY}\right] + y^{(2)}_{XY}\left[-2\tfrac{M^2}{\beta^2}\psi^{(1)}_{X} + 2x^{(1)}_{Y}\right] + \\
&+ y^{(2)}_{XX}\left[-3(1-\tfrac{M^2}{\beta^2})x^{(1)}_{X} + 3\tfrac{M^2}{\beta^2}\psi^{(1)}_{Y} - \psi^{(1)}_{Y}\right] + y^{(2)}_{YY}\left[-5x^{(1)}_{X} - \psi^{(1)}_{Y}\right] + \\
&+ \psi^{(1)}_{XX}\Big\{2y^{(2)}_{Y} - 7(1-\tfrac{M^2}{\beta^2})(x^{(1)}_{X})^2 + (x^{(1)}_{Y})^2 - 6\tfrac{M^2}{\beta^2}\psi^{(1)}_{Y}x^{(1)}_{X} + \\
&\qquad + \tfrac{M^2}{\beta^2}\left[(1-\tfrac{M^2}{\beta^2})(\psi^{(1)}_{X})^2 - (\psi^{(1)}_{Y})^2\right]\Big\} + \\
&+ \psi^{(1)}_{XY}\left[-2y^{(2)}_{X} + 2\tfrac{M^2}{\beta^2}\psi^{(1)}_{X}\psi^{(1)}_{Y} + 6\tfrac{M^2}{\beta^2}\psi^{(1)}_{X}x^{(1)}_{X} - 8x^{(1)}_{X}x^{(1)}_{Y}\right] + \\
&+ x^{(1)}_{XX}\left[-2(1-\tfrac{M^2}{\beta^2})x^{(1)}_{X}\psi^{(1)}_{X} + 2\tfrac{M^2}{\beta^2}\psi^{(1)}_{Y}\psi^{(1)}_{X} + (1-\tfrac{M^2}{\beta^2})y^{(2)}_{X}\right] + \\
&+ x^{(1)}_{YY}\left[-4x^{(1)}_{X}\psi^{(1)}_{X} + y^{(2)}_{X}\right] + x^{(1)}_{XY}\left[-2\tfrac{M^2}{\beta^2}(\psi^{(1)}_{X})^2 + 2x^{(1)}_{Y}\psi^{(1)}_{X}\right].
\end{aligned}
\tag{3.47}
$$

Equations (3.45), (3.46), and (3.47) are to be used to investigate the possible improvement in the solution by a constant shift in the coordinate system; that is $x^{(2)}$ = constant. In these equations the order of the derivatives of $\psi^{(1)}$ near the leading edge is found from the expression

$$
\psi^{(1)} = kY - \frac{1}{\sqrt{2}}\sqrt{-X^2 + k^2Y^2 + 1 + \sqrt{1-2(X^2-Y^2k^2) + (X^2+k^2Y^2)^2}}
\tag{3.48}
$$

which is the solution to (3.45).

Near $Y \to 0$ one finds:

$$\begin{aligned}
\Psi^{(1)}_{Y} &\to k \\
\Psi^{(1)}_{X} &\to \frac{X}{(X^2-1)^{1/2}} \\
\Psi^{(1)}_{XY} &\to 0 \\
\Psi^{(1)}_{XX} &\to \frac{1}{(X^2-1)^{3/2}} \\
\Psi^{(1)}_{YY} &\to -\frac{k^2}{(X^2-1)^{3/2}}
\end{aligned} \tag{3.49}$$

Now since the correction $x^{(2)}$ = constant is under consideration, and since $x^{(1)}$, the correction of order ε, is to be zero on the boundary, then as before we take $x^{(1)} \equiv 0$ and (3.46) becomes an equation for $y^{(2)}$,

$$\left(1-\frac{M^2}{\beta^2}\right)y^{(2)}_{XX}+y^{(2)}_{YY} = 2\frac{M^2}{\beta^2}\left[\Psi^{(1)}_{X}\Psi^{(1)}_{XY} - \Psi^{(1)}_{Y}\Psi^{(1)}_{XX}\right]. \tag{3.50}$$

This is the usual equation obtained in the ordinary perturbation method for the second order correction. Near the leading edge equations (3.49) show that the nonhomogenous term has the behavior

$$\Psi^{(1)}_{Y}\Psi^{(1)}_{XX} \simeq \frac{1}{(X^2-1)^{3/2}} \quad \text{or, for } X^2-1 \simeq \varepsilon^2,$$

$$\simeq \frac{1}{\varepsilon^3}. \tag{3.51}$$

Neglecting the solution for $y^{(2)}$ for the moment, the substitution

$$x^{(2)} = \text{constant} \tag{3.52}$$

into equation (3.47) gives

$$(1-\frac{M^2}{\beta^2})\, y^{(3)}_{XX} + y^{(3)}_{YY} = y^{(2)}_{XY}(-2\frac{M^2}{\beta^2}\psi^{(1)}_X) + y^{(2)}_{XX}(3\frac{M^2}{\beta^2}-1)\psi^{(1)}_Y -$$

$$- y^{(2)}_{YY}\psi^{(1)}_Y +$$

$$+ \psi^{(1)}_{XX}\left\{ 2y^{(2)}_Y + \frac{M^2}{\beta^2}\left[(1-\frac{M^2}{\beta^2})(\psi^{(1)}_X)^2 - (\psi^{(1)}_Y)^2\right]\right\}$$

$$+ \psi^{(1)}_{XY}\left[-2y^{(2)}_X + 2\frac{M^2}{\beta^2}\psi^{(1)}_X\psi^{(1)}_Y\right]. \qquad (3.53)$$

In this equation the term $(\psi^{(1)}_X)^2\psi^{(1)}_{XX}$ near the leading edge has the behavior $\frac{1}{(X^2-1)^{5/2}} \simeq \frac{1}{\varepsilon^5}$ and thus equation (3.47) becomes of equal importance with equation (3.45) in this region since

$$\varepsilon^3 \text{ times order } \frac{1}{\varepsilon^5} = \varepsilon \text{ times order } \frac{1}{\varepsilon^3} = \text{order } \frac{1}{\varepsilon^2}. \qquad (3.54)$$

In other words the possibility of setting $x^{(2)}$ equal to a constant has been denied since it leads to requiring a solution of equation (3.53), and it is not possible to effect a simple correction in the compressible case.*

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* In this connection we mention that although the second order corrections to the velocities in the usual perturbation solutions are no more singular near the leading edge then the first order terms for the case of incompressible flow (c.f. p. 201 [11]), for the compressible case an increase in singularity does occur at the second order level (p. 325 [8]).

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Phyllis Fox was born in Denver, Colorado, on March 13, 1923. She attended Denver schools, and then Wellesley College in Massachusetts from which she was graduated in 1944 with a B.A. degree in mathematics. From 1944 to 1946 she worked as an engineering assistant associated with the differential analyzer of the General Electric Company in Schenectady, New York. She then returned to Colorado to obtain a B.S. degree in electrical engineering from the University of Colorado in 1948. In the fall of that year she entered M. I. T. and in 1949 received her master's degree in electrical engineering with a thesis concerned with the solution of electric power networks on large-scale digital computers. During most of her doctoral study she was a research assistant on a program of research on Machine Methods of Computation supported by the Office of Naval Research. She has taken a job with the Computation Center of New York University's Institute for Mathematics and Mechanics.